October 19, 2006 **Name** ______________________________

The total number of points available is 148. Throughout this test, **show your work.**

1. (9 points) Let $f(x) = x^3 - 2x - 3$.
   (a) Compute $f'(x)$
   Solution: $f'(x) = 3x^2 - 2$
   (b) What is $f'(2)$?
   Solution: $f'(2) = 3 \cdot 2^2 - 2 = 10$.
   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(2, f(2))$.
   Solution: Since $f(2) = 2^3 - 2 \cdot 2 - 3 = 1$, using the point-slope form leads to $y - 1 = f'(2)(x - 2) = 10(x - 2)$, so $y = 10x - 19$.

2. (12 points) Consider the function f defined by:

$$f(x) = \begin{cases} x + x^3 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 2x^{1/2} & \text{if } x > 1 \end{cases}$$

   (a) Is f continuous at $x = 1$?
   Solution: Yes, the limits from the left and right are both 2, and the value of f at 1 is 2.
   (b) What is the slope of the line tangent to the graph of f at the point $(4, 4)$?
   Solution: To find $f'(4)$ first note that when x is near 8, $f(x) = 2x^{1/2}$ so $f'(x) = 2\frac{1}{2}x^{-1/2}$. Thus, $f'(4) = 2\frac{1}{2}4^{-1/2} = 2\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$.
   (c) Find $f'(-3)$
   Solution: To find $f'(-3)$, we must differentiate the part of f defined for $x < 1$. In this area, $f'(x) = 1 + 3x^2$, so $f'(-3) = 1 + 3(-3)^2 = 28$.

3. (15 points) If a ball is thrown vertically upward from the roof of 212 foot building with a velocity of 48 ft/sec, its height after t seconds is $s(t) = 212 + 48t - 16t^2$.

   (a) What is the height the ball at time $t = 0$?
   Solution: $s(0) = 212$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?
   Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) At what time is the velocity zero?
   Solution: Solve $48 - 32t = 0$ to get $t = 3/2$.

   (d) What is the maximum height the ball reaches?
   Solution: Solve $s'(t) = 48 - 32t = 0$ to get $t = 3/2$ when the ball reaches its zenith. Thus, the max height is $s(3/2) = 212 + 48(3/2) - 16(3/2)^2 = 248$.

   (e) What is the velocity of the ball when it hits the ground (height 0)?
   Solution: Solve $s(t) = 0$ using the quadratic formula to get $t = \frac{3 \pm \sqrt{9+52}}{2} = \frac{3 \pm \sqrt{61}}{2} \approx 5.405$, since the larger is the only reasonable answer. Find $s'(5.405) \approx -124.96$ feet/sec.

4. (10 points) The cost of producing x units of stuffed alligator toys is $C(x) = -0.003x^2 + 6x + 6000$ for $0 \leq x \leq 1000$.

   (a) Find the marginal cost at the production level of 1000 units.
   Solution: $C'(x) = \frac{d}{dx} - 0.003x^2 + 6x + 6000 = -0.006x + 6$ so $C'(1000) = -6 + 6 = 0$.

   (b) Find the (incremental) cost of producing the 1000^{th} toy.
   Solution: $C(1000) - C(999) = -0.003(1000 - 999)^2 + 6(1000 - 999) + 6000 - 6000 = -0.003(1999) + 6 = 0.003$.

5. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = f(x) + g(x)$. Compute $L'(2)$.
Solution: $L'(x) = f'(x) + g'(x)$, so $L'(2) = f'(2) + g'(2) = 4 + 4 = 8$.

(b) Let $U(x) = g \circ g(x)$. Compute $U(1)$.
Solution: $U(1) = g(g(1)) = g(2) = 3$.

(c) Let $K(x) = g(x^2) \cdot f(x)$. Compute $K(2)$.
Solution: $K(2) = g(4) \cdot f(2) = 2 \cdot 6 = 12$.

(d) Again, $K(x) = g(x^2) \cdot f(x)$. Compute $K'(2)$.
Solution: $K'(x) = g'(x^2) \cdot 2x \cdot f(x) + f'(x)g(x^2)$, so $K'(2) = g'(4) \cdot 4 \cdot f(2) + f'(2)g(4) = 6 \cdot 4 \cdot 6 + 4 \cdot 2 = 152$.

(e) Let $V(x) = f(f(x))$. Compute $V'(3)$.
Solution: Again, by the chain rule, $V'(x) = f'(f(x)) \cdot f'(x) \cdot 2$, so $V'(3) = f'(f(3)) \cdot f'(3) \cdot 2 = f'(1) \cdot f'(3) = 6 \cdot 2 = 12$.

(f) Let $W(x) = g(2x) \div f(x)$. Compute $W'(1)$.
Solution: By the quotient rule, $W'(x) = [g'(2x) \cdot 2 \cdot f(x) - g(2x) \cdot f'(x)] \div (f(x))^2$ so $W'(1) = [g'(2) \cdot 2 \cdot f(1) - g(2) \cdot f'(1)] \div (f(1))^2 = (4 \cdot 2 \cdot 4 - 3 \cdot 6) \div 16 = (32 - 18)/16 = 7/8$.

(g) Let $Z(x) = f(x^2 + g(x))$. Compute $Z'(1)$.
Solution: Again by the chain rule, $Z'(x) = f'(x^2+g(x)) \cdot \frac{d}{dx}(x^2+g(x)) = f'(x^2 + g(x)) \cdot (2x + g'(x))$, so $Z'(1) = f'(1 + g(1)) \cdot (2 + g'(1)) = f'(3) \cdot (2 + 5) = 2 \cdot 7 = 14$.

6. (25 points) Compute the following derivatives. There is no need to simplify except in part (c).

   (a) Let $f(x) = (x + \sqrt{1 + x^3})$. Find $\frac{d}{dx} f(x)$.

   Solution: Note that $\sqrt{x^3} = x^{3/2}$, so we differentiate it using the power rule and chain rule: $f'(x) = 1 + \frac{1}{2}(1 + x^3)^{-1/2} \cdot 3x^2 = 1 + \frac{3x^2}{2\sqrt{1+x^3}}$.

   (b) Let $g(x) = x^3/\sqrt{1 + x^2}$. What is $g'(x)$?

   Solution: Use the quotient rule to get $g'(x) = [3x^2\sqrt{1 + x^2} - \frac{1}{2}(1 + x^2)^{-1/2}(x^3)] \div (1 + x^2) = \frac{3x^2\sqrt{1+x^2} - x^4/\sqrt{1+x^2}}{1+x^2}$.

   (c) Find $\frac{d}{dx}((x + 2) \cdot (2x - 1)^2)$.

   Solution: By the product rule, $\frac{d}{dx}((x+2) \cdot (2x - 1)^2) = 1 \cdot 2(2x - 1)^2) + 2(2x - 1) \cdot 2(x + 2) = 12x^2 + 8x - 7 = (2x - 1)(6x + 7)$.

   (d) Find $\frac{d}{dx}\sqrt{\frac{2x+1}{3x^2-2}}$.

   Solution: By the chain and quotient rules, $\frac{d}{dx}\frac{2x + 1}{3x^2 - 2} = \frac{1}{2}\left(\frac{2x + 1}{3x^2 - 2}\right)^{-1/2} \cdot \frac{2(3x^2 - 2) - 6x(2x + 1)}{(3x^2 - 2)^2}$.

   (e) Find $\frac{d}{dt}(t - 1/t^2)^3$.

   Solution: By the chain rule, $\frac{d}{dt}(t - 1/t^2)^3 = 3(t - 1/t^2)^2 \cdot (1 + 2t^{-3})$.

7. (40 points) Consider the function

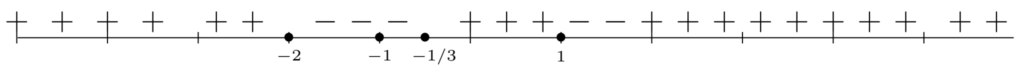
$$f(x) = \sqrt{\frac{(x^2-1)(3x+1)}{(2x^2-8)(x+1)}}.$$

Use the Test Interval Technique to find the (implied) domain of $f(x)$.

Solution: Let $r(x)$ denote the part inside the radical. So we need to solve the inequality $r(x) \geq 0$. Notice first that r is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$r(x) = \frac{(x-1)(x+1)(3x+1)}{2(x-2)(x+2)(x+1)}.$$

We can cancel the common factors $x+1$ with the understanding that we are (very slightly by including -1) enlarging the domain of r: $r(x) = \frac{(x-1)(3x+1)}{2(x-2)(x+2)}$. Next find the branch points. These are the points at which r can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $-2, -1/3, 1, 2$. Again we select test points and find the sign of r at of these points to get the sign chart.



Recall that we are solving $r(x) \geq 0$. The solution to $r(x) > 0$ is easy. It is the union of the open intervals with the + signs, $(-\infty, -2) \cup (-1/3, 1) \cup (2, \infty)$. It remains to solve $r(x) = 0$ and attach these solutions to what we have. The zeros of r are 1 and $-1/3$. So the solution to $r(x) \geq 0$ is $(-\infty, -2) \cup [-1/3, 1] \cup (2, \infty)$. Notice that the branch point -1 is not included since r is not defined at -1. Thus, the domain of the function $f(x)$ is $(-\infty, -2) \cup [-1/3, 1] \cup (2, \infty)$.

8. (7 points) Suppose $f(x)$ satisfies $f(3) = 2$ and the line tangent to the graph of f at the point $(3, 2)$ is $2y + 3x = 13$. What is $f'(3)$?

Solution: The slope of the given line is $-3/2$, so $f'(3) = -3/2$.