March 5, 2004 **Name** ______________________________

The total number of points available is 120. Throughout this test, **show your work.**

1. (15 points) Let $p(x) = x^2 - 4x + 5$.

   (a) Compute $p'(x)$

   Solution: $p'(x) = 2x - 4$.

   (b) Compute $p''(x)$

   Solution: $p''(x) = 2$

   (c) Use the information in a. to find an equation for the line tangent to the graph of p at the point $(1, 2)$.

   Solution: $y - 2 = p'(1)(x - 1) = -2(x - 1)$, so $y = -2x + 4$.

2. (15 points) Intermediate Value Theorem. Recall that the IVT asserts the following: If f is a continuous function on the interval $[a, b]$ and M is a number between $f(a)$ and $f(b)$, then there exists a number c satisfying $a \leq c \leq b$ and $f(c) = M$. For this problem let $f(x) = \sqrt{4x - 3}$ and let $[a, b] = [1, 7]$. Finally, suppose $M = 2$. Find the number c whose existence is guaranteed by IVT.

   Solution: We need to solve the equation $\sqrt{4x - 3} = 2$ for x. Square both sides to get $4x - 3 = 4$, from which it follows that $x = 7/4$.

3. (15 points) The total weekly cost in dollars incurred by the Lincoln Record Company in pressing x playing records is given by $C(x) = 2000 + 3x - 0.01x^2$ for x in the range 0 to 6000.

   (a) Find the marginal cost function $C'(x)$.
   Solution: $C'(x) = 3 - 0.02x$

   (b) Find the average cost function $\overline{C}(x)$.
   Solution: $\overline{C}(x) = \frac{2000+3x-0.01x^2}{x} = \frac{2000}{x} + 3 - 0.01x$.

   (c) Find the marginal average cost function $\overline{C}'(x)$.
   Solution: $\overline{C}'(x) = -2000x^{-2} - 0.01$.

   (d) Interpret your results in (c). Is the average cost growing or falling as the company produces more units?
   Solution: The function $\overline{C}'(x) = -2000x^{-2} - 0.01$ is negative throughout its domain. This means that the average cost decreases the more records are produced.

4. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	1	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $K(x) = f(x) \cdot g(x)$. Compute $K'(2)$

Solution: $K'(x) = f'(x) \cdot g(x) + g'(x)f(x)$, so $K'(2) = f'(2) \cdot g(2) + g'(2)f(2) = 4\dot{3} + 4\dot{6} = 36$.

(b) Let $L(x) = f(x)/g(x)$. Compute $L'(1)$.

Solution: $L'(x) = (f'(x)g(x) - g'(x)f(x)) \div (g(x))^2$, so $L'(1) = (f'(1)g(1) - g'(1)f(1)) \div (g(1))^2 = (6 \cdot 1 - 5 \cdot 4) \div 1^2 = -14$.

(c) Let $U(x) = f \circ g(x)$. Compute $U'(4)$.

Solution: By the chain rule, $U'(x) = f'(g(x)) \cdot g'(x)$, so $U'(4) = f'(g(4)) \cdot g'(4) = f'(2) \cdot g'(4) = 4 \cdot 6 = 24$.

(d) Let $V(x) = g \circ f(x)$. Compute $V'(5)$.

Solution: Again, by the chain rule, $V'(x) = g'(f(x)) \cdot f'(x)$, so $V'(5) = g'(f(5)) \cdot f'(5) = g'(5) \cdot f'(5) = 1 \cdot 3 = 3$.

(e) Let $W(x) = f(x^2)$. Compute $W'(2)$.

Solution: Again by the chain rule, $W'(x) = f'(x^2) \cdot 2x$, so $W'(2) = f'(4) \cdot 4 = 5 \cdot 4 = 20$

(f) Let $Z(x) = g(xf(x))$. Compute $Z'(3)$.

Solution: Again by the chain rule and the product rule, $Z'(x) = g'(xf(x)) \cdot \frac{d}{dx}xf(x) = g'(xf(x)) \cdot (1 \cdot f(x) + xf'(x))$, so $Z'(3) = g'(3f(3)) \cdot (1 \cdot f(3) + 3f'(3)) = g'(3 \cdot 1) \cdot (1 \cdot 1 + 3 \cdot 2) = 3(1 + 6) = 21$.

5. (25 points) Compute the following derivatives.

   (a) Let $f(x) = x^2 + x^{-\frac{1}{2}}$. Find $\frac{d}{dx}f(x)$.
   Solution: $f'(x) = 2x - \frac{1}{2}x^{-\frac{3}{2}}$.

   (b) Let $g(x) = \sqrt{x^4 + 4}$. What is $g'(x)$?
   Solution: $g'(x) = \frac{1}{2}(x^4 + 4)^{-\frac{1}{2}} \cdot 4x^3 = 2x^3 \div \sqrt{x^4 + 4}$.

   (c) Find $\frac{d}{dx}((3x + 1)^2 \cdot (4x^3 - 1))$.
   Solution: By the product rule, $\frac{d}{dx}((3x + 1)^2 \cdot (4x^3 - 1)) = 2(3x + 1) \cdot 3 \cdot (4x^3 - 1) + 12x^2(3x + 1)^2 = 6(3x + 1)(10x^3 + 2x^2 - 1)$.

   (d) Find $\frac{d}{dx}\frac{2x^2+1}{x-2}$.
   Solution: By the quotient rule, $\frac{d}{dx}\frac{2x^2+1}{x-2} = \frac{4x(x-2)-1(2x^2+1)}{(x-2)^2} = \frac{2x^2-8x-1}{x^2-4x+4}$.

   (e) Find $\frac{d}{dt}(t^2 + 1/t)^3$.
   Solution: By the chain rule, $\frac{d}{dt}(t^2 + 1/t)^3 = 3(t^2 + 1/t)^2 \cdot (2t - t^{-2})$.

6. (10 points) Let $f(x) = 2/x$. Use the limit definition of derivative to find $f'(x)$.

Solution: We need to compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Thus,

$$\begin{aligned} \lim_{h\to 0} \frac{2/(x+h) - 2/x}{h} &= \lim_{h\to 0} \frac{2/(x+h) - 2/x}{h} \\ &= \lim_{h\to 0} \frac{2x - 2(x+h)}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{-2h}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{-2}{x(x+h)} \\ &= \frac{-2}{x^2} \end{aligned}$$

7. (10 points) Consider the function f defined by:

$$f(x) = \begin{cases} 2x^2 - 3x & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ \sqrt{x-1} & \text{if } x > 1 \end{cases}$$

(a) What is the slope of the line tangent to the graph of f at the point $(5, 2)$?

Solution: To find $f'(5)$ first note that when x is near 5, $f(x) = (x-1)^{1/2}$ so $f'(x) = \frac{1}{2}(x-1)^{-1/2}$. Thus, $f'(5) = \frac{1}{2}(4^{-1/2}) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$.

(b) Find $f'(-3)$

Solution: To find $f'(-3)$, we must differentiate the part of f defined for $x < 1$. In this area, $f'(x) = 4x - 3$, so $f'(-3) = 4(-3) - 3 = -15$