November 2, 2016 **Name** ______________________________

The problems count as marked. The total number of points available is 166. Throughout this test, **show your work.** This is an amalgamation of the tests from sections 3 and 10.

1. (30 points) Let $p(x) = 2x^3 + 3x^2 - 12x$

   (a) Find an equation for the line tangent to $p(x)$ at the point $(1, p(1))$.

   Solution: Since $p'(x) = 6x^2 + 6x - 12 = 6(x-1)(x+2)$, $p'(1) = 0$ and the line through $(1, -7)$ with slope 0 is $y = -7$.

   (b) Find an interval over which $p(x)$ is decreasing.

   Solution: The two critical points are $x = 1$ and $x = -2$, and $p'(x) < 0$ in the interval $[-2, 1]$, so p is decreasing on $[-2, 1]$.

   (c) Find an interval over which $p(x)$ is concave upwards.

   Solution: Since $p''(x) = 12x + 6$, it follows that p is concave upwards on $(-1/2, \infty)$.

2. (12 points) Let $f(x) = \sqrt{x^2 - x + 3}$.

   (a) Compute $f'(x)$
   Solution: $f'(x) = \frac{1}{2}(x^2 - x + 3)^{-1/2} \cdot (2x - 1) = \frac{2x-1}{2\sqrt{x^2-x+3}}$.

   (b) What is $f'(3)$?
   Solution: $f'(3) = \frac{2 \cdot 3 - 1}{2 \cdot 9^{1/2}} = 5/6$

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(3, f(3))$.
   Solution: Since $f(3) = 3$, using the point-slope form leads to $y - 3 = f'(3)(x - 3) = 5(x - 3)/6$, so $y = 5x/6 + 1/2$.

3. (16 points) Polynomials f, g, and h have degrees 4, 5, and 6 respectively. For each polynomial below, find the degree. The symbol $\circ$ means composition.

   (a) $f \cdot g + g \cdot h + h \cdot f$ ________________
   Solution: 11

   (b) $f \circ (g + h)$ ________________
   Solution: 24

   (c) $f \circ (g \circ h)$ ________________
   Solution: 120

   (d) $f \circ (g \cdot h)$ ________________
   Solution: 44

4. (30 points) Let $H(x) = (x^2-9)^2(3x+1)^3$.

   (a) Use the chain and product rules to find $H'(x)$.

   Solution:

$$\begin{aligned} H'(x) &= 2(x^2-9)\cdot 2x(3x+1)^3 + 3(3x+1)^2 \cdot 3(x^2-9)^2 \\ &= (x^2-9)(3x+1)^2[4x(3x+1)+9(x^2-9)] \\ &= (x-3)(x+3)(3x+1)^2[21x^2+4x-81] \end{aligned}$$

   (b) Find the critical points of H.

   Solution: Thus, H has critical points $x = \pm 3, -1/3$ and $\frac{-4\pm\sqrt{6820}}{42}$. The last two are roughly $\alpha \approx -2.06$ and $\beta \approx 1.87$.

   (c) Build the sign chart for $H'(x)$

   Solution: $H'(x) \geq 0$ on $(-\infty, -3] \cup [\alpha, \beta] \cup [3, \infty)$.

   (d) Classify the critical points of H as max, min, or imposters.

   Solution: Notice that $-1/3$ is an imposter since $h'(x)$ is positive on both sides of $-1/3$. Then we have maximums at -3 and β and minimums at α and 3.

   (e) Find the intervals over which H is increasing.

   Solution: From the sign chart for H', we see that H is increasing on $(-\infty, -3] \cup [\alpha, \beta] \cup [3, \infty)$.

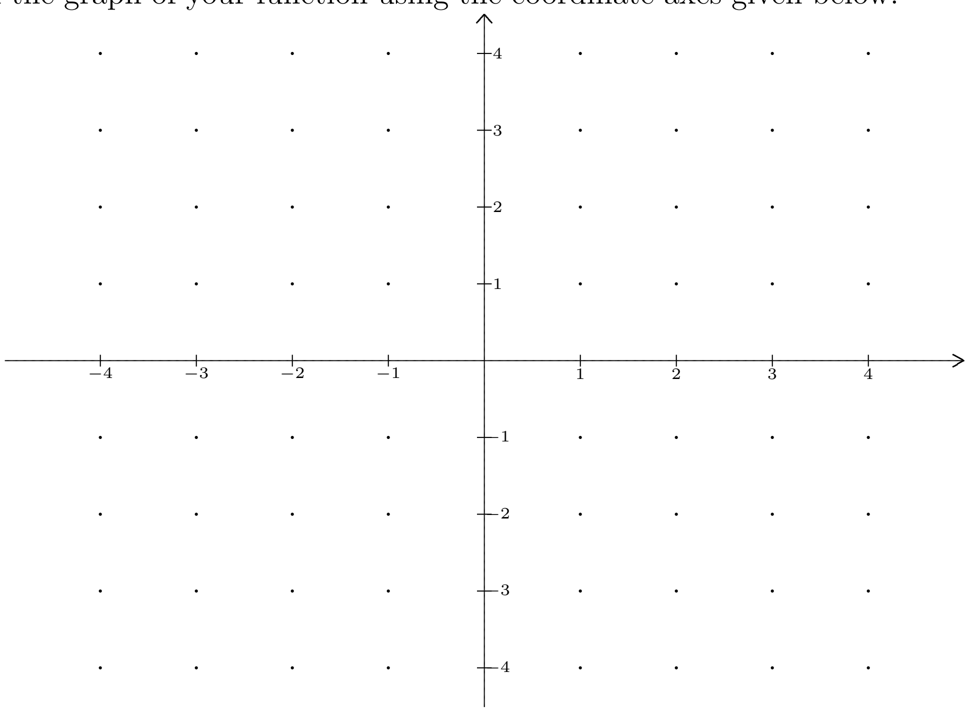
5. (30 points) Build a rational function $r(x)$ with the following properties:

- r has vertical asymptotes at $x = -1$ and $x = 1$.
- r has zeros at $x = -2$ and $x = 0$.
- r satisfies $\lim_{x\to\infty} r(x) = 2$.

(a) Find a symbolic representation for r.
Solution: One such function is $r(x) = \frac{2(x+2)x}{(x+1)(x-1)}$.

(b) Build the sign chart for your function.
Solution: Our r is positive on $(-\infty, -2) \cup (-1, 0) \cup (1, \infty)$.

(c) Sketch the graph of your function using the coordinate axes given below.



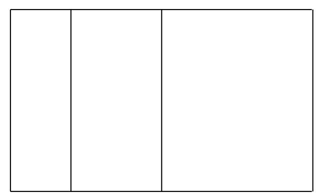
(d) Find $r'(x)$ for your r.
Solution: Using the quotient rule, we have $r'(x) = \frac{(4x+4)(x^2-1)-2x(2x^2+4x)}{(x^2-1)^2}$.

(e) Does r have any critical points? If so, find them.
Solution: The numerator of r' reduces to $-4x^2 - 4x - 4 = -4(x^2 + x + 1)$ which has no zeros, so r has no critical points.

6. (12 points) Mike thinks of a positive number. He adds the square and the reciprocal of his number. What is the least possible sum Mike could get?

   Solution: Let x denote Mike's number. Then $S(x) = x^2 + 1/x$ is the sum of the square and the reciprocal of Mike's number. So $S'(x) = 2x - x^{-2}$, which is zero at $x = 2^{-1/3}$, so we get $S(2^{-1/3} = 2^{-2/3} + 2^{1/3} \approx 0.629 + 1.25 \approx 1.88$.

7. (12 points) A farmer has 20000 feet of fencing to build a rectangular pasture. But he must separate the goats, horses and cows into different parts of the pasture using two vertical straight sections of fence as shown.



   What is the area of the largest pasture the farmer can build?

   Solution: Label the vertical pieces x and the horizontal pieces y. Then $2y + 4x = 20000$. To maximize the area, write $A = xy = x(5000 - x/2)$. Then $A'(x) = -x + 5000$ which has a zero at $x = 5000$. We can see that this maximizes A, and the maximum value os $A = 5000 \cdot 2500 = 12500000$.

8. (24 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	3	4	2	3
3	1	2	5	3
4	3	5	2	5
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = f(2x) \cdot g(x)$. What is $L(2)$? Compute $L'(2)$. Find an equation for the line tangent to L at the point $(2, L(2))$.

Solution: $L(2) = f(4)g(2) = 3 \cdot 2 = 6$. $L'(x) = 2f'(2x)g(x) + f(2x)g'(x)$, so $L'(2) = 2f'(2)g(2) + f(2)g'(2) = 20 + 9 = 29$. The tangent line is $y - 6 = 29(x - 2)$.

(b) Let $U(x) = \sqrt{g(x)}$. Compute $U(5)$ and $U'(5)$. Is the function U increasing or decreasing at $x = 5$?

Solution: $U(5) = \sqrt{g(5)} = 2$. On the other hand $U'(x) = \frac{1}{2}(g(x)^{-1/2} \cdot g'(x)$ by the chain rule, so $U'(5)$ has the value $\frac{1}{2}4^{-1} \cdot g'(5) = 1/4$. Since $U'(5)$ is positive, U is increasing at $x = 5$.

(c) Let $K(x) = (g(x) + f(x))^2$. Compute $K(1)$ and $K'(1)$

Solution: $K(1) = (g(1) + f(1))^2 = (2 + 3)^2 = 25$. $K'(x) = 2(g(x) + f(x))(g'(x) + f'(x))$, so $K'(1) = 2(g(1) + f(1))(g'(1) + f'(1)) = 2(6)(11) = 132$.