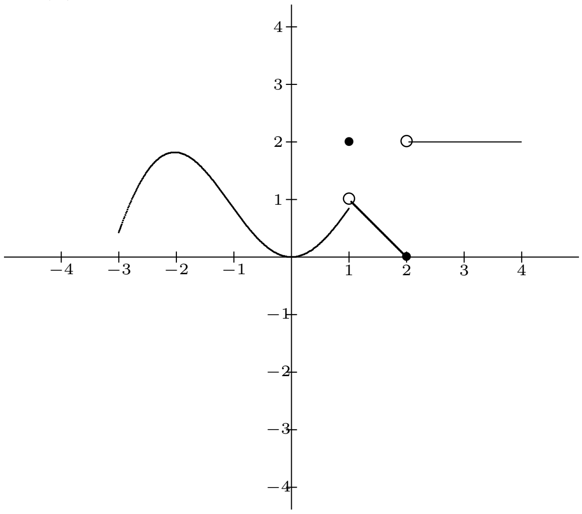
October 18, 2001 Your name ____________________________

The multiple choice problems count 4 points each. In the multiple choice section, circle the correct choice (or choices). You must show your work on the other problems 5 through 10. The total number of points available is 131.

1. Questions (a) through (e) refer to the graph of the function f given below.



(a) $\lim_{x \to 3} f(x) =$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** does not exist

Solution: $\lim_{x \to 3} f(x) = 2$, by the blotter test.

(b) $\lim_{x \to 2^-} f(x) =$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** does not exist

Solution: $\lim_{x \to 2^-} f(x) = 0$, by the blotter test.

(c) A good estimate of $f'(-2)$ is

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** there is no good estimate

Solution: $f'(-2) = 0$ because the tangent line is horizontal.

(d) A good estimate of $f'(-1)$ is

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** there is no good estimate

Solution: $f'(-1) = -1$ because the tangent line has a negative slope.

(e) A good estimate of $f'(3)$ is

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** there is no good estimate

Solution: $f'(3) = 0$ because the tangent line is horizontal.

2. The line tangent to the graph of a function f at the point $(2, 3)$ on the graph also goes through the point $(-1, 9)$. What is $f'(2)$?

   (A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

   Solution: The slope must be $m = \frac{9-3}{-1-2} = -2$.

3. What is the slope of the tangent line to the graph of $f(x) = (3x)^{-1}$ at the point (1,1/3)?

   (A) $-2/9$ **(B)** $-1/3$ **(C)** $-2/27$ **(D)** $-2/81$ **(E)** 0

   Solution: First note that $f'(x) = -1(3x)^{-2} \cdot 3 = -x^{-2}/3$ so $f'(1) = -1/3$.

4. True-false questions. These count 2 points each.

   (a) True or false. If $f'(x) > 0$ for each x in the interval $(-1, 1)$, then f is increasing on $(-1, 1)$.

   Solution: True.

   (b) True or false. If $f(a) < 0, f(b) > 0$, and $f'(x) > 0$ for each x in (a, b), then there is one and only one number c in (a, b) such that $f(c) = 0$.

   Solution: True. The Intermediate Value Theorem guarantees that there is at least one c in (a, b), and the condition $f'(x) > 0$ for each x in (a, b) guarantees that there can be no more than 1 such point.

   (c) True or false. The graph of a function cannot touch or intersect a horizontal asymptote to the graph of f.

   Solution: False. There is nothing in the definition of horizontal asymptote that implies this.

   (d) True or false. If $f'(c) = 0$, then f has a relative maximum or a relative minimum at $x = c$.

   Solution: False. The function can have neither a max nor a min at a stationary point. Look at $f(x) = x^3$ and 0.

   (e) True or false. If f has a relative maximum or a relative minimum at $x = c$, then $f'(c) = 0$.

   Solution: False. All we can tell is that c is a critical point. It might be a singular point.

(f) True or false. If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at $x = c$.

Solution: True. This is just the second derivative test.

(g) True or false. If f and g are differentiable, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.

Solution: False. Look up the product rule.

(h) True or false. If f and g are differentiable, then $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)}{g'(x)}$.

Solution: False. Look up the quotient rule.

(i) True or false. If f and g are differentiable and $h(x) = f \circ g$, then $h'(x) = f[g(x)]g'(x)$.

Solution: False. Look up the chain rule.

(j) True or false. If f and g are differentiable and a and b are constants, then $\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$.

Solution: True. This is just the rule that talks about the derivative of the sum and of a constant times a function.

On all the following questions, **show your work.**

5. (10 points) Let $f(x) = 1/(2x)$.

(a) Construct $\frac{f(2+h)-f(2)}{h}$

Solution: $\frac{f(2+h)-f(2)}{h} = \frac{1/2(2+h)-1/2(2)}{h} = \frac{(4-(4+2h))/4(4+2h)}{h} = \frac{-2h/4(4+2h)}{h}$.

(b) Simplify and take the limit of the expression in (a) as h approaches 0 to find $f'(2)$.

Solution: $\lim_{h\to 0} \frac{-2h/4(4+2h)}{h} = \lim_{h\to 0} \frac{-2}{4(4+2h)} = -1/8.$

(c) Use the information found in (b) to find an equation for the line tangent to the graph of f at the point $(2, 1/4)$.

Solution: The equation for the line is $y - 1/4 = -1/8(x - 2)$ which in slope-intercept is $y = -x/8 + 1/2$.

6. (8 points) Suppose $f(x)$ is a function such that $f(2) = 1$ and $f'(x) = 3x + 4$ for all real numbers x. Let L denote the line that is tangent to the graph of $f(x)$ at the point $(2, 1)$. What is the slope of L? What is the y-intercept of L? What is the x-intercept of L?

Solution: The slope of the line is $f'(2) = 3 \cdot 2 + 4 = 10$, and the line itself is $y - 1 = 10(x - 2)$ which is just $y = 10x - 19$. The y-intercept is -19 and the x-intercept is 1.9.

7. (8 points) Find an equation of the tangent line to the graph of $f(x) = \sqrt{2x-5}$ at the point (3,1).

   Solution: First differentiate f to get $f'(x) = \frac{1}{2}(2x-5)^{-1/2} \cdot 2 = (2x-5)^{-1/2}$. Thus $f'(3) = 1$, and the line is given by $y - 1 = 1(x-3)$.

8. (15 points)

   (a) State the hypothesis of the Intermediate Value Theorem (IVT).

   Solution: The function f must be continuous over an interval $[a,b]$, and the number M is between $f(a)$ and $f(b)$.

   (b) State the conclusion of the Intermediate Value Theorem.

   Solution: Then there is a number c in the open interval (a,b) such that $f(c) = M$.

   (c) Does the function $f(x) = \sqrt{x+4}$ satisfy the hypothesis of IVT over the interval $[0,12]$. If so, find a whole number M between $f(0)$ and $f(12)$, and then find a number c in the interval $(0,12)$ such that $f(c) = M$.

   Solution: Note that $f(0) = \sqrt{4} = 2$ and $f(12) = \sqrt{16} = 4$, and the only whole number between these two is 3. So we need to solve the equation $f(c) = \sqrt{c+4} = 3$. We can do this by squaring both sides to get $c+4 = 9$ which means $c = 5$.

9. (12 points) Suppose the functions f and g and their derivatives are given by the table of values shown. Complete the table by calculating the values of the derivatives of both $f \circ g(x)$ and $g \circ f(x)$ for each of the values of x in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\frac{df\circ g(x)}{dx}$	$\frac{dg\circ f(x)}{dx}$
0	2	3	1	3		
1	3	4	5	2		
2	2	1	1	4		
3	5	3	4	1		
4	4	1	3	2		
5	2	0	0	4		

Solution: First note that the notation $\frac{df\circ g(x)}{dx}$ means $(f \circ g)'(x)$, ie, the derivative of the composition of f and g. Thus, we use the chain rule. The sixth entry of the top row, for example, is $f'(g(0) \cdot g'(0) = f'(1) \cdot g'(0) = 4 \cdot 3 = 12$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\frac{df\circ g(x)}{dx}$	$\frac{dg\circ f(x)}{dx}$
0	2	3	1	3	12	12
1	3	4	5	2	0	4
2	2	1	1	4	16	4
3	5	3	4	1	1	12
4	4	1	3	2	6	2
5	2	0	0	4	12	0

10. (30 points) Compute the following derivatives.

(a) Let $f(x) = x^2 - (1/x)$. Find $\frac{d}{dx}f(x)$.
Solution: $f'(x) = 2x + x^{-2}$.

(b) Let $g(x) = \sqrt{3x^3 + 4}$. What is $g'(x)$?
Solution: $g'(x) = \frac{1}{2}(3x^3 + 4)^{-1/2} \cdot 9x^2 = \frac{9x^2}{2\sqrt{3x^3+4}}$.

(c) Find $\frac{d}{dx}((2x + 1)^3 \cdot (3x^2 - 1))$
Solution: The derivative is obtained by the product rule: $3(2x+1)^2 \cdot 2 \cdot (3x^2 - 1) + (2x + 1)^3 \cdot 6x = 6(2x + 1)^2(5x^2 + x - 1)$.

(d) Find $\frac{d}{dx}\frac{2x+1}{x^2+2}$
Solution: By the quotient rule, the derivative is $\frac{2(x^2+2)-(2x+1)2x}{(x^2+2)^2} = \frac{-2x^2-2x+4}{(x^2+2)^2}$.

(e) Find $\frac{d}{dt}(t^{-3} + t^{-2})^3$.
Solution: By the chain rule, the derivative is $3(t^{-3}+t^{-2})^2(-3t^{-4}-2t^{-3})$.