November 4, 2015 **Name** ____________________

The problems count as marked. The total number of points available is 147. Throughout this test, **show your work.** Use of calculator to circumvent ideas discussed in class will generally result in no credit.

1. (20 points) Demonstrate your understanding of the product, quotient and chain rules by differentiating each of the given functions. No need to simplify. You must show your work.

   (a) Let $F(x) = (x^2 - 3x + 1)(x^3 - 2x + 5)$
   Solution: Note that $F'(x) = (2x-3)(x^3-2x+5)+(3x^2-2)(x^2-3x+1)$.

   (b) $G(x) = \dfrac{2x^4 - 3x + 1}{x^2 - x + 3}$
   Solution: By the quotient rule, $G'(x) = \frac{(8x^3-3)(x^2-x+3)-(2x-1)(2x^4-3x+1)}{(x^2-x+3)^2}$.

   (c) $K(x) = (x^2 - 3)^{17}$
   Solution: By the chain rule, $K'(x) = 17(x^2 - 3)^{16} \cdot 2x = 34x(x^2 - 3)^{16}$.

   (d) $H(x) = \sqrt{(3x + 1)^4 - 7}$.
   Solution: By the chain rule, $H'(x) = \frac{1}{2}((3x+1)^4-7)^{-1/2} \cdot 4(3x+1)^3 \cdot 3 = \frac{6(3x+1)^3}{\sqrt{(3x+1)^4-7}}$.

2. (12 points) Mike thinks of a positive number. He adds the square of his number and the reciprocal of his number. What is the smallest possible sum he could obtain? A calculator solution will get no credit. Estimate the answer to the nearest 0.01.

   Solution: We want to minimize the function $f(x) = x^2 + \frac{1}{x}$. We can do this by noting that $f'(x) = 2x - 1/x^2$ which is zero at just one place, $x = (1/2)^{1/3}$. Taking $f(1/2^{1/3}) \approx 1.889 \approx 1.89$.

3. (10 points) The line tangent to the graph of $g(x)$ at the point $(4, 7)$ has a y-intercept of 9. What is $g'(4)$?

   Solution: The line has slope $(9 - 7)/(0 - 4) = -1/2$.

4. (10 points) Find a point on the graph of $h(x) = x^3 - 6x^2 + 9x$ where the tangent line is horizontal. There are two such points on the graph of $h(x)$.

   Solution: Since $h'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$, it follows that h has a horizontal tangent line for $x = 1$ and $x = 3$. The points on the graph are $(1, 4)$ and $(3, 0)$.

5. (10 points) Suppose f and g are functions for which both $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Which of the following could be true? Circle all the options that could be true.

   (A) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 0$ **(B)** $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ does not exist **(C)** $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = -1$

   (D) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 7$ **(E)** $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 1$

   Solution: We know this limit can be any real number. For example, let r be any number and let $f(x) = r(x - a)$ and $g(x) = x - a$. Then

   $$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} r = r.$$

6. (35 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = (f(x) + g(x))^2$. Compute $L(2)$ and $L'(2)$.

Solution: $L(2) = 81$ and $L'(x) = 2(f(x) + g(x))(f'(x) + g'(x))$, so $L'(2) = 2(f(2) + g(2))(f'(2) + g'(2)) = 2(6 + 3)(4 + 4) = 144$.

(b) Let $U(x) = f \circ f \circ f(x)$. Compute $U(1)$ and $U'(1)$.

Solution: First, $U(1) = f(f(f(1))) = f(3) = 1$. By the chain rule, $U'(x) = f'(f \circ f(x)) \cdot f'((f(x)) \cdot f'(x)$, so $U'(1) = f'(f \circ f(1)) \cdot f'((f(1)) \cdot f'(1) = f'(f((f(1)) \cdot f'(f(1)) \cdot f'(1) = f'(3) \cdot f'(4) \cdot f'(1) = 2 \cdot 5 \cdot 6 = 60$.

(c) Let $K(x) = g(x) + f(x^2)$. Compute $K(2)$ and $K'(2)$

Solution: $K(2) = g(2) + f(4) = 3 + 3 = 6$ and $K'(x) = g'(x) + f'(x^2)2x$, so $K'(2) = g'(2) + f'(2^2)2 \cdot 2 = 4 + 5 \cdot 4 = 24$.

(d) Let $Z(x) = 1/g(2x)$. Compute $Z(3)$ and $Z'(3)$.

Solution: $Z(3) = 1/g(6) = 1/2$. Rewriting Z as $Z(x) = g(2x)^{-1}$, by the chain rule, we have $Z'(x) = -1g(2x)^{-2} \cdot g'(2x) \cdot 2$ so $Z'(3) = -1 \cdot (1/2)^{-2}(g(6))^2 \cdot 2 = -2$.

(e) Let $Q(x) = g(3x) + f(2x)$. Compute $Q(2)$ and $Q'(2)$.

Solution: First, $Q(2) = g(6) + f(4) = 2 + 3 = 5$. By the sum rule and chain rule, $Q'(x) = 3g'(3x) + 2f'(2x)$ so $Q'(2) = 3g'(6) + 2f'(4) = 3 \cdot 4 + 2 \cdot 5 = 22$.

7. (20 points) Consider the fourth degree polynomial $p(x) = (x^2 - 4)^2$.

   (a) Find the intervals over which $p(x)$ is increasing.

   Solution: First find the derivative: $p'(x) = 4x(x^2-4) = 4x(x-2)(x+2)$, which is positive on $[-2, 0] \cup [2, \infty)$. So p is increasing over these two intervals.

   (b) Find the intervals over which $p(x)$ is concave upwards.

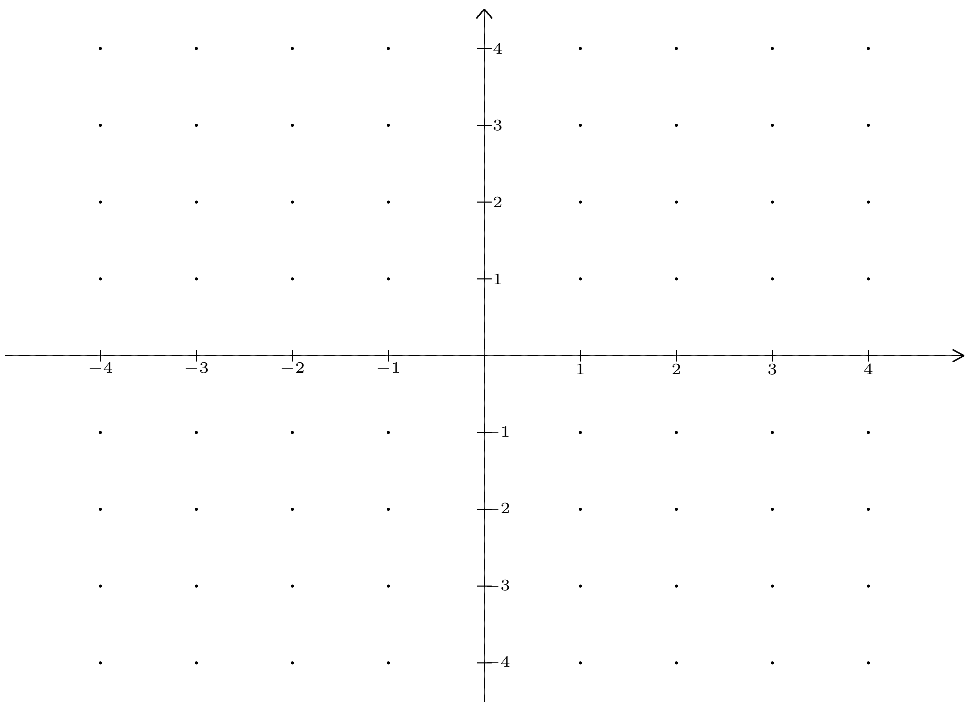
   Solution: Note that $p'(x) = 4x(x^2-4)$ and $p''(x) = 4(x^2-4)+4x(2x) = 12x^2 - 16$. Solving $p''(x) \geq 0$, we find that p is concave upwards on both $(-\infty, -\frac{2\sqrt{3}}{3})$ and $(\frac{2\sqrt{3}}{3}, \infty)$.

8. (30 points) Consider the function

$$r(x) = \frac{(x^2-4)(6x)}{(3x-6)(x+1)(x-3)}.$$

Use the Test Interval Technique to find the sign chart of $r(x)$. Find the zeros and the horizontal and vertical asymptotes, and sketch the graph of r. Your graph must be consistent with the information you find in the sign chart.

Solution:



Solution: Notice first that r is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$r(x) = \frac{(x-2)(x+2)(6x)}{3(x-2)(x-3)(x+1)}.$$

We can remove the common factor $x-2$ with the understanding that we are (very slightly) enlarging the domain of r: $r(x) = \frac{(x+2)(6x)}{3(x-3)(x+1)}$. Next find the branch points. These are the points at which r can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $0, -1, -2, 3$. The horizontal asymptote is $y = 6/3 = 2$, the vertical asymptotes are $x = 3$ and $x = -1$ and the zeros of r are $x = 0$ and $x = -2$. Again we select test points and find the sign of f at of these points to get the sign chart.