October 28, 2014 **Name** ______________________________

The problems count as marked. The total number of points available is 145. Throughout this test, **show your work.**

1. (30 points) Let $f(x) = 3x^4 + 4x^3 - 72x^2$.

   (a) Find the critical points of f.

   Solution: First note that $f'(x) = 12x^3 + 12x^2 - 144x$, which factors into $12x(x+4)(x-3)$, so the critical points of f are $0, -4$ and 3.

   (b) Build the sign chart for $f'(x)$.

   Solution: First note that $f'(x) = 12x^3 + 12x^2 - 144x$, which factors into $12x(x+4)(x-3)$, so we can build the sign chart for f' using the branch points $0, -4$ and 3. Doing this yields the result that f' is negative over each of the intervals $(-\infty, -4)$ and $(0, 3)$. Therefore f is decreasing over these two intervals.

   (c) Use this information in part (b) to find the intervals over which f is increasing.

   Solution: First note that $f'(x) = 12x^3 + 12x^2 - 144x$, which factors into $12x(x+4)(x-3)$, so we can build the sign chart for f' using the branch points $0, -4$ and 3. Doing this yields the result that f' is negative over each of the intervals $(-\infty, -4)$ and $(0, 3)$. Therefore f is decreasing over these two intervals.

   (d) Discuss the concavity of f.

   Solution: To discuss the concavity of f, we must build the sign chart for $f''(x)$. $f''(x) = 36x^2 + 24x - 144$. Using the quadratic formula, we find that $f''(x) = 0$ has two roots, $x = \frac{-2\pm\sqrt{148}}{6}$. So $x \approx 1.69, x \approx -2.36$. Since the graph of f'' is a parabola which opens upwards, we can say that f is concave downward between these two values and upwards on the other two intervals.

   (e) Find $f(1)$ and $f'(1)$. Use this information to find the line tangent to f at $(1, f(1))$ in slope-intercept form.

   Solution: First note that $f(1) = -63$. Since $f'(x) = 12x^3 + 12x^2 - 144x$, it follows that $f'(1) = -120$. So the tangent line is $y + 63 = -120(x-1) = -120x + 120$, so the line in slope-intercept form is $y = -120x + 57$.

2. (35 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = 2f(x) \cdot g(x)$. Compute $L'(5)$.

Solution: $L'(x) = 2(f'(x)g(x) + g'(x)f(x))$, so $L'(5) = 2(f'(5)g(5) + g'(5)f(5)) = 2(3 \cdot 4 + 1 \cdot 5) = 34$.

(b) Let $U(x) = f(3x) \div g(2x)$. Compute $U(2)$ and $U'(2)$.

Solution: $U(2) = f(6) \div g(4) = 0$. By the chain and quotient rules, $U'(x) = \frac{f'(3x)\cdot 3 \cdot g(2x) - g'(2x)\cdot 2 \cdot f(3x)}{(g(2x))^2}$, so $U'(2) = \frac{f'(6)\cdot 3\cdot g(4) - g'(4)\cdot 2\cdot f(6)}{(g(4))^2} = \frac{3\cdot 3\cdot 2 - 6\cdot 2\cdot 0}{4} = \frac{18-0}{4} = 9/2$.

(c) Let $K(x) = g(x + f(x))$. Compute $K(3)$ and $K'(3)$.

Solution: $K(2) = g(4) + f(2) = 2 + 6 = 8$ and $K'(2) = 2g'(4) + f'(2) = 12 + 4 = 16$.

(d) Let $V(x) = f(g(f(x)))$. Compute $V'(3)$.

Solution: Again, by the chain rule, $V'(x) = f'(g(2x)) \cdot g'(2x) \cdot 2$, so $V'(3) = f'(g(6)) \cdot g'(6) \cdot 2 = f'(2) \cdot g'(6) \cdot 2 = 4 \cdot 4 \cdot 2 = 32$.

(e) Let $W(x) = g(x^2 - 1)$. Compute $W'(2)$.

Solution: By the chain rule, $W'(x) = g'(x^2 - 1) \cdot 2x$ so $W'(2) = g'(3) \cdot 2 \cdot 2 = 3 \cdot 4 = 12$.

3. (20 points) Recall that $\frac{d}{dx}e^{g(x)} = e^{g(x)} \cdot g'(x)$. Find the intervals over which the function $f(x) = x^2e^{2x}$ is increasing. Write your answer in interval notation.

   Solution: Use the product rule to get $f'(x) = 2xe^{2x} + x^2e^{2x} \cdot 2 = e^{2x}(2x^2 + 2x) = e^{2x} \cdot 2x \cdot (x+1)$. Build the sign chart for f' to learn that $f'(x)$ is negative only when $0 < x < 1$. So f is increasing on both $(-\infty, -1]$ and $[0, \infty)$.

4. (15 points) Two positive numbers x and y are related by $2x + 3y = 16$. What is the largest possible product xy could be, and what pair (x, y) achieves that product? Note that if $y = 2$, then $x = 5$ and the product $xy = 10$. If $y = 4$, then $x = 2$ and the product is 8.

   Solution: Solve $2x + 3y = 16$ for y to get $f(x) = xy = x(\frac{16-2x}{3}) = \frac{16x-2x^2}{3}$. So $f'(x) = (16 - 4x)/3$ and $x = 4$ is the only critical point. So $x = 4$ and $y = 8/3$. It follows that the maximum value of xy is $4 \cdot 8/3 = 32/3$.

5. (15 points) Two positive numbers x and y are related by $xy = 10$. What is the smallest possible value $6x + 3y$ could have?

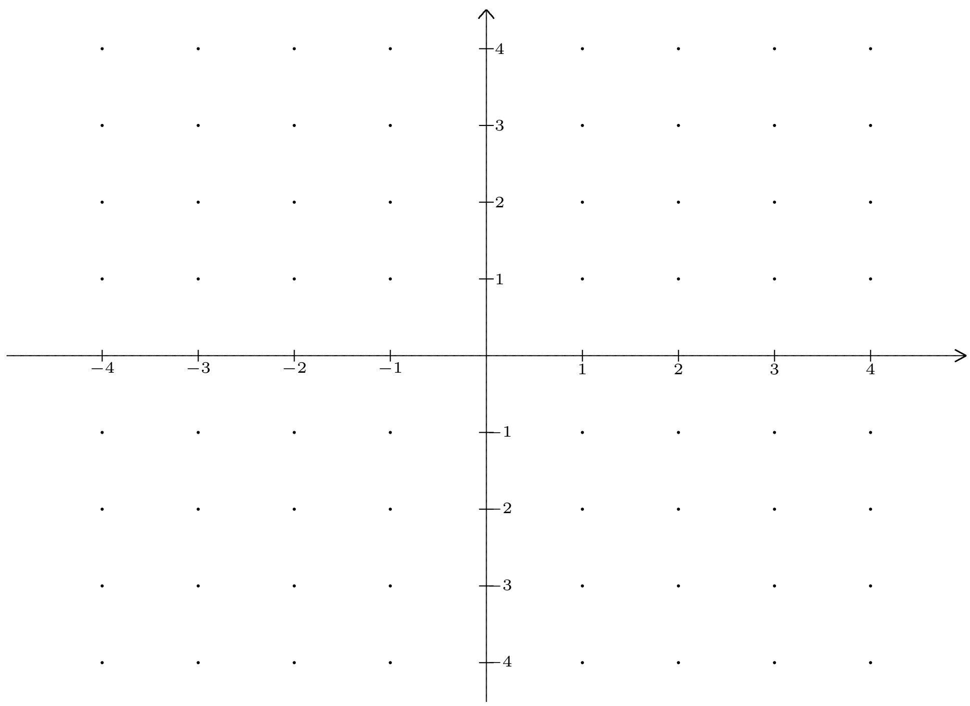
   Solution: Note that $y = 10/x$, so the sum to be maximized is $f(x) = 6x + 3 \cdot 10/x$. Taking the derivative, we have $f'(x) = 6 - 30/x^2$ which is zero when $x = \sqrt{5}$. Looking at the sign chart for f', we see that f' is negative on the left of $\sqrt{5}$ and positive to the right of it. Therefore the smallest possible value of $6x + 3y$ is $6\sqrt{5} + 3 \cdot 2\sqrt{5} = 12\sqrt{5}$.

6. (30 points) Consider the function

$$r(x) = \frac{(x^2-4)(6x)}{(3x-6)(x+1)(x-3)}.$$

Use the Test Interval Technique to find the sign chart of $r(x)$. Find the zeros and the horizontal and vertical asymptotes, and sketch the graph of r. Your graph must be consistent with the information you find in the sign chart.

Solution:



Solution: Notice first that r is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$r(x) = \frac{(x-2)(x+2)(6x)}{3(x-2)(x-3)(x+1)}.$$

We can remove the common factor $x-2$ with the understanding that we are (very slightly) enlarging the domain of r: $r(x) = \frac{(x+2)(6x)}{3(x-3)(x+1)}$. Next find the branch points. These are the points at which r can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $0, -1, -2, 3$. The horizontal asymptote is $y = 6/3 = 2$, the vertical asymptotes are $x = 3$ and $x = -1$ and the zeros of r are $x = 0$ and $x = -2$. Again we select test points and find the sign of f at of these points to get the sign chart.