March 14, 2014 **Name** ________________________

The problems count as marked. The total number of points available is 174. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (24 points) Demonstrate your understanding of the product, quotient and chain rules by differentiating each of the given functions. Find the critical points for each function and the intervals over which the function is increasing. You must show your work.

   (a) Let $F(x) = (2x - 8)(3x + 6)$.
   Solution: Note that $F'(x) = 2(3x + 6) + 3(2x - 8) = 12 - 12$, so the only critical point is $x = 1$. Since $F'(x) > 0$ on $(1, \infty)$, we conclude that F is increasing on that interval.

   (b) Let $G(x) = (x^2 - 4)^3$.
   Solution: $G'(x) = 3(x^2 - 4)^2(2x)$, so $x = \pm 2, x = 0$ are critical points. The sign chart confirms that G is increasing on $[0, 2)$ and $[2, \infty)$. Since G is also continuous on its domain, we conclude that G is increasing on $[0, \infty)$.

   (c) Let $H(x) = \frac{2x-3}{x-4}$.
   Solution: $H'(x) = \frac{2(x-4)-(2x-3)}{(x-4)^2} = \frac{-5}{(x-4)^2}$, so G has no critical points. Since $G'(x) < 0$ at all the points of its domain (all real numbers except 4), we conclude that G is decreasing over both $(-\infty, 4)$ and $(4, \infty)$.

2. (24 points)

   (a) Build a cubic polynomial $p(x)$ with zeros at $x = -2$, $x = 0$ and $x = 4$.

   Solution: One way yo do this is to take the product of the three linear factors $x + 2, x$ and $x - 4$ to get $p(x) = x^3 - 2x^2 - 8x$.

   (b) Find the critical points of p.

   Solution: We have $p'(x) = 3x^2 - 4x - 8$. We can use the quadratic formula to get the roots of $p'(x) = 0$: $x = \frac{4 \pm \sqrt{112}}{6} = \frac{2 \pm 2\sqrt{7}}{3}$.

   (c) Find the interval(s) over which the polynomial p is decreasing.

   Solution: The sign chart shows that p is decreasing between the two zeros of p', that is on $[\frac{2-2\sqrt{7}}{3}, \frac{2+2\sqrt{7}}{3}]$.

3. (12 points) Mike thinks of a positive number. He adds the square of his number and the reciprocal of his number. What is the smallest possible sum he could obtain? A calculator solution will get not credit. Estimate the answer to the nearest 0.01.

   Solution: We want to minimize the function $f(x) = x^2 + \frac{1}{x}$. We can do this by noting that $f'(x) = 2x - 1/x^2$ which is zero at just one place, $x = (1/2)^{1/3}$. Taking $f(1/2^{1/3}) \approx 1.889 \approx 1.89$.

4. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	1	4

(a) $Q(x) = g(x)/f(x)$. Find $Q(5)$ and $Q'(5)$.

Solution: First, $Q(5) = g(5)/f(5) = 4/5$. $Q'(x) = (g'(x)f(x) - f'(x)g(x))/(f(x))^2$. Therefore, $Q'(5) = (g'(5)f(5) - f'(5)g(5))/(f(5))^2 = \frac{1\cdot5-3\cdot4}{5^2} = \frac{-7}{25}$.

(b) Let $H(x) = f(x^2+1)$. Compute $H(1)$ and $H'(1)$.

Solution: $H(1) = f(2) = 6$. By the chain rule, $H'(x) = f'(x^2+1)\cdot 2x$. Therefore, $H'(1) = f'(2)\cdot 2 = 4\cdot 2 = 8$.

(c) Let $W(x) = [g(x)]^3 + f(x)$. Compute $W(4)$ and $W'(4)$.

Solution: $W(4) = g(4)^3 + f(4) = 8 + 3 = 11$. Again by the chain rule, $W'(x) = 3g(x)^2 \cdot g'(x) + f'(x)$, so $W'(4) = 3g(4)^2 \cdot g'(4) + f'(4) = 3\cdot 2^2\cdot 6 + 5 = 77$.

(d) Let $L(x) = (x^2+1)\cdot f(x)$. Compute $L(2)$ and $L'(2)$.

Solution: First, $L(2) = 5\cdot f(2) = 5\cdot 6 = 30$. By the product rule, $L'(x) = 2xf(x) + (x^2+1)f'(x)$, so $L'(2) = 4\cdot 6 + 5\cdot 4 = 44$.

(e) Let $U(x) = f\circ g\circ f(x)$. Compute $U(6)$ and $U'(6)$.

Solution: First, $U(6) = f\circ g\circ f(2) = f(g(0)) = f(3) = 1$ By the chain rule, $U'(x) = f'(g(f(x)))\cdot g'(f(x))\cdot f'(x)$, so $U'(6) = f'(3)\cdot 2\cdot 3 = 2\cdot 2\cdot 3 = 12$.

(f) Let $Z(x) = g(2x+1)\cdot f(3x-1)$. Compute $Z(1)$ and $Z'(1)$.

Solution: Note that $Z(1) = g(3)\cdot f(2) = 5\cdot 6 = 30$. Again by the chain rule and the product rule, $Z'(x) = g'(2x+1)\cdot 2\cdot f(3x-1) + 3f'(2)\cdot g(3) = 3\cdot 2\cdot 6 + 3\cdot 4\cdot 5 = 96$.

5. (20 points) Consider the line L given by $y = 2x$, the point $P = (-4, 2)$, and the circle whose equation is $x^2 - 8x + y^2 - 4y = -16$.

   (a) Find the point on the line that is closest to the point P.

   Solution: The slope of the line L is 2 so the slope of the line joining P with the point on L closest to P is $-1/2$. The line through P with slope $-1/2$ goes through the origin, a point of L.

   (b) Find the point on the circle that is closest to the point P.

   Solution: The circle has center $(4, 2)$ and the radius is 2 which we find by completing the square. So the point of the circle closest to P is $(2, 2)$

   (c) Find the point on the line that is closest to the circle.

   Solution: The line through the circles center with slope $-1/2$ is given by $y - 2 = -\frac{1}{2}(x - 4)$, and this line hits L at $(8/5, 16/5)$.

6. (24 points) Let

$$f(x) = \begin{cases} |x+2| & \text{if } -\infty < x \leq 0 \\ \sqrt{2x+2} & \text{if } 0 < x \leq 4 \\ x^{2/3} & \text{if } 4 < x < \infty \end{cases},$$

(a) Find an equation for the line tangent to the graph of f at the point $(-3, 1)$.

Solution: Since $m = -1$, we have $y - 1 = -1(x + 3)$.

(b) Find an equation for the line tangent to the graph of f at the point $(1, 2)$.

Solution: Since $f'(x) = \frac{1}{2}(2x + 2)^{-1/2} \cdot 2$ near $x = 1$, it follows that $f'(1) = 1/2$ and the tangent line is $y - 2 = 1/2(x - 1)$.

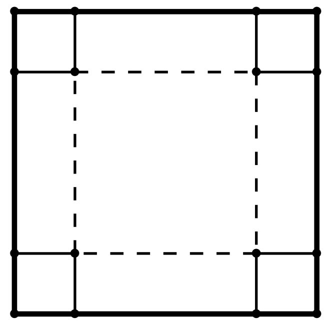
(c) Find an equation for the line tangent to the graph of f at the point $(8, 4)$.

Solution: Since $f'(x) = \frac{2}{3}x^{-1/3}$ near $x = 8$, it follows that $f'(8) = 2/3 \cdot 1/2 = 1/3$ and the tangent line is $y - 4 = (1/3)(x - 8)$.

(d) Discuss the continuity of f at the point $x = 0$.

Solution: The left and right limits of f at 0 are different, so f cannot be continuous there.

7. (20 points) We are given a 8in $\times$ 8in sheet of paper as shown. Squares of size $x \times x$ are cut from each corner and then the sides are folded upward along the dashed lines. Let $V(x)$ denote the volume of the topless box obtained. (This is the same problem as you had in Set 5).



(a) What is the natural domain of V? Calculate $V(1)$ and $V(2)$.
Solution: $V(1) = 6 \cdot 6 \cdot 1 = 36$ and $V(2) = 4 \cdot 4 \cdot 2 = 32$.

(b) Sketch the graph of $V(x)$ over its natural domain.
Solution: $V(x) = (8 - 2x)^2 \cdot x$ is a cubic polynomial with zeros at $x = 0$ and $x = 4$.

(c) Find the maximum value of $V(x)$.
Solution: Differentiate V to get $V'(x) = 12x^2 - 64x + 64 = 4(3x^2 - 16x + 16)$ which has zeros at $x = 4/3$ and $x = 4$. Of course $V(4) = 0$ so the maximum value of V is $V(4/3) = 1024/27 \approx 37.93$.

8. (20 points) If a ball is thrown vertically upward from the roof of 256 foot building with a velocity of 64 ft/sec, its height after t seconds is $s(t) = 256 + 64t - 16t^2$.

   (a) What is the height the ball at time $t = 1$?

   Solution: $s(1) = 256 + 64 - 16 = 304$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) At what time t does the ball reach its maximum height?

   Solution: Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith.

   (d) What is the maximum height the ball reaches?

   Solution: Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith. Thus, the max height is $s(2) = 256 + 64(2) - 16(2)^2 = 320$.

   (e) After how many seconds is the ball exactly 176 feet above the ground?

   Solution: Use the quadratic formula to solve $256+64t-16t^2 = 176$. You get $-16t^2 + 64t + 80 = 0$. The left side can be factored to get solutions $x = -1$(nonsense) and $x = 5$(yes!).

   (f) How fast is the ball going when it reaches the height 176? Write the answer with the correct units.

   Solution: Using the solution $t = 5$ above, we find that the velocity at $t = 5$ is $v(5) = 64 - 32 \cdot 5 = -96$ feet per second.

   (g) How fast is the ball going when it hits the ground?

   Solution: The ball hits the ground when $s(t) = 0$ which, by the quadratic formula is $t = 2 \pm 2\sqrt{5}$. Of course, only the positive on of these is relevant. Now $v(2 + 2\sqrt{5}) = 64 - 32(2 + 2\sqrt{5}) = -64\sqrt{5} \approx -143$ feet per second. Its negative because the ball is moving downward.

   (h) The second derivative $s''(t)$ of the position function, also called the *acceleration* function, is denoted $a(t)$. Compute $a(t)$. Explain why this function is negative for all values of t.

   Solution: $a(t) = s''(t) = \frac{d}{dt}v(t) = \frac{d}{dt}64-32t = -32$. Its negative because the force of gravity pulls downward.