October 3, 2013 **Name** ______________________________

The problems count as marked. The total number of points available is 171. Throughout this test, for full credit you must **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (6 points) Find an equation in slope-intercept form for a line parallel to the line $3x - 6y = 7$ which goes through the point $(-3, 5)$.

   Solution: The given line has slope 1/2 so the one parallel has slope 1/2 also. Hence $y - 5 = (1/2)(x + 3)$. Thus $y = x/2 + 13/2$.

2. (10 points) What is the smallest possible value of the expression

$$|x - 1| + |x - 2| + |x - 4|?$$

   Solution: Experiment with the values $x = 1, 2$ and 4 to see that at $x = 2$, the expression has value $1 + 0 + 2 = 3$ and that for all x between 2 and 4, the value is $x - 1 + x - 2 + 4 - x = x + 1 \geq 3$ while for values of x between 1 and 2, we have $x - 1 + 2 - x + 4 - x = 5 - x \geq 3$. So we conclude that 3 is the least possible value.

3. (10 points) The set of points satisfying $(x-1)^2 + (y-2)^2 = 16$ is a circle. The set of points satisfying $x^2 + 4x + y^2 + 6y = 100$ is also a circle. What is the slope of the line connecting the centers of the two circles?

   Solution: The center of the first is $(1, 2)$ and the center of the second is $(-2, -3)$, so the slope of the line is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2+3}{1+2} = \frac{5}{3}$.

4. (35 points) Evaluate each of the limits (and function values) indicated below.

   (a) $\lim\limits_{x\to 6} \dfrac{\sqrt{2x-3}-3}{x-6}$

   Solution: Rationalize the numerator to get $\lim\limits_{x\to 6} \dfrac{\sqrt{2x-3}-3}{x-6} =$
   $\lim\limits_{x\to 6} \dfrac{(\sqrt{2x-3}-3)(\sqrt{2x-3}+3)}{(x-6)(\sqrt{2x-3}+3)} =$
   $\lim\limits_{x\to 6} \dfrac{(2x-3)-9}{(x-6)(\sqrt{2x-3}+3)} = \lim\limits_{x\to 6} \dfrac{2(x-6)}{(x-6)(\sqrt{2x-3}+3)} = 1/3$

   (b) $\lim\limits_{x\to 2} \dfrac{3x-6}{\frac{1}{2x}-\frac{1}{4}}$

   Solution: The limit of both the numerator and the denominator is 0, so we must do the fractional arithmetic. The limit becomes

$$\lim_{x\to 2} \frac{3x-6}{\frac{2-x}{4x}} = \lim_{x\to 2} \frac{3(x-2)}{-\frac{x-2}{4x}} = \lim_{x\to 2} \frac{3}{-\frac{1}{4x}} = -24.$$

   (c) $\lim\limits_{x\to 3} \dfrac{x^3-3x^2}{x^2-2x-3}$

   Solution: Factor both numerator and denominator to get $\lim\limits_{x\to 3} \dfrac{(x-3)x^2}{(x-3)(x+1)} =$
   $\lim\limits_{x\to 3} \dfrac{x^2}{x+1} = 9/4$

   (d) $\lim\limits_{x\to \infty} \dfrac{(2x-3)^3}{x(4x-1)^2}$

   Solution: Both numerator and denominator are polynomials of degree 3, so we simply need to take the quotient of the coefficients of x^3. Thus, we have lim $= 8/16 = 1/2$.

   (e) $\lim\limits_{x\to 0} \dfrac{(x+1)^3-1}{x}$

   Solution: Expand the numerator to get $\lim_{x\to 0} \frac{x^3+3x^2+3x+1-1}{x} = \lim_{x\to 0} \frac{x^3+3x^2+3x}{x} =$
   $\lim_{x\to 0} \frac{x(x^2+3x+3)}{x} = 3$

5. (30 points)

The following ten problems are worth 3 points each. For problems (a) through (j), let

$$f(x) = \begin{cases} 2x+1 & \text{if } -3 \le x < -1 \\ 3x-1 & \text{if } -1 \le x \le 2 \\ x+3 & \text{if } 2 < x \le 4 \\ 1 & \text{if } 4 < x \le 6 \end{cases}$$

Find the value, if it exists, of each item below. Use DNE when the value does not exist.

(a) What is the domain of the function f. Express your answer in interval notation.

Solution: $D = [-3, 6]$.

(b) $\lim_{x\to -1^-} f(x)$

Solution: -1

(c) $\lim_{x\to -1^+} f(x)$

Solution: -4

(d) $\lim_{x\to -1} f(x)$

Solution: DNE because the left limit and right limit are different.

(e) $f(-1)$

Solution: -4

(f) $\lim_{x\to 2^-} f(x)$

Solution: 5

(g) $\lim_{x\to 2^+} f(x)$

Solution: 5

(h) $\lim_{x\to 2} f(x)$

Solution: 5, because both the left limit and the right limit are 5.

(i) $f(2)$

Solution: 5

(j) $\lim_{x\to 4} f(x)$

Solution: This limit does not exist.

6. (15 points) Let $H(x) = (\sqrt{x^2-1}-2)^3$.

   (a) What is the (implied) domain of H?

   Solution: The values of x that make the number in the radical negative are $(-1, 1)$, so the domain is $(-\infty, -1] \cup [1, \infty)$.

   (b) Find five functions, f, g, h, l, and k so that $H(x) = f \circ g \circ h \circ l \circ k(x)$.

   Solution: One set that works is $k(x) = x^2, l(x) = x-1, h(x) = \sqrt{x}, g(x) = x - 2$, and $f(x) = x^3$.

   (c) Compute $H'(x)$.

   Solution: By the chain rule, $H'(x) = 3(\sqrt{x^2-1}-2)^2 \cdot \frac{d}{dx}(\sqrt{x^2-1}-2) = 3(\sqrt{x^2-1}-2)^2 \cdot \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x$.

7. (10 points) If $g(x) = (x^2-1)^2(2x+1)^3$, then

$$g'(x) = 4x(x^2-1)(2x+1)^3 + 6(x^2-1)^2(2x+1)^2.$$

Find all the x-intercepts of the function $g'(x)$.

Solution: Factor out the common terms to get $g'(x) = (x^2-1)(2x+1)^2[4x((2x+1) + 6(x^2-1)] = (x^2-1)((2x+1)^2[14x^2+4x-6]$. Setting each factor equal to zero, we find the zeros are $x = \pm 1, x = -1/2$ and $x = \frac{-2\pm\sqrt{88}}{14}$.

8. (20 points) Let $f(x) = \sqrt{3x+1}$. Notice that $f(5) = \sqrt{3 \cdot 5 + 1} = 4$.

   (a) Find the slope of the line joining the two points $(4, f(4))$ and $(5, f(5))$.
   Solution: The slope is $\frac{f(5)-f(4)}{5-4} = \frac{4-\sqrt{13}}{1} \approx 0.394$.

   (b) Let h be a positive number. What is the slope of the line passing through the points $(5, f(5))$ and $(5+h, f(5+h))$. Your answer depends on h of course.
   Solution: $\frac{f(5+h)-f(5)}{h} = \frac{\sqrt{3(5+h)+1}-\sqrt{3 \cdot 5+1}}{h} = \frac{\sqrt{3h+16}-\sqrt{16}}{h}$.

   (c) Compute $\lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$ to get $f'(5)$.
   Solution: Since we get zero over zero we recall that, in this case, we should rationalize the numerator.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(5+h)-f(5)}{h} &= \lim_{h\to 0} \frac{\sqrt{3(5+h)+1}-\sqrt{3\cdot 5+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(5+h)+1}-\sqrt{16}}{h} \cdot \frac{\sqrt{15+3h+1}+4}{\sqrt{15+3h+1}+4} \\
&= \lim_{h\to 0} \frac{16+3h-16}{h(\sqrt{3(5+h)+1}+4)} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(5+h)+1}+4)} \\
&= \lim_{h\to 0} \frac{3}{(\sqrt{3(5+h)+1}+4)} \\
&= \frac{3}{2(4)} \\
&= \frac{3}{8}
\end{aligned}$$

   So, $f'(5) = 3/8$.

   (d) Your answer to (c) is the slope of the line tangent to the graph of f at the point $(5, f(5))$. In other words, your answer is $f'(5)$. Write and equation for the tangent line.
   Solution: The line is $y - 4 = \frac{3}{8}(x-5)$, or $y = 3x/8 + 17/8$.

9. (20 points) Let $G(x) = \sqrt{(x-4)(2x+1)(x+3)(x+5)}$

   (a) Find the domain of G and express it as a union of intervals (in interval notation).

   Solution: Build the sign chart for $F(x) = (x-4)(2x+1)(x+3)(x+5)$ and use it to solve $(x-4)(2x+1)(x+3)(x+5) \geq 0$. You get $(-\infty, -5) \cup (-3, -1/2) \cup (4, \infty)$.

   (b) You might have used $x = 5$ as a test point in part a. On the other hand you might have used $x = 6$. Given that the function $F(x) = (x-4)(2x+1)(x+3)(x+5)$ is continuous over the real numbers, explain why the Intermediate Value Theorem guarantees that the sign of $F(5)$ is the same as the sign of $F(6)$.

   Solution: If $F(5)$ and $F(6)$ had different signs, the IVT would imply that somewhere between 5 and 6 the function F had a zero, but we have found all the zeros of F.

10. (15 points) Find a (symbolic representation for a) quadratic polynomial whose graph includes the points $(-1, 0), (3, -16)$ and $(5, 0)$.

    Solution: $p(x) = 2x^2 - 8x - 10$ works. One way to get this is to note that $f(x) = a(x+1)(x-5)$ has the two zeros $x = -1$ and $x = 5$. Thus we choose a so that $f(3) = -16$. But $f(3) = a \cdot 4 \cdot (-2) = -8a$, so $a = 2$.