October 4, 2011 **Name** ______________________________

The problems count as marked. The total number of points available is 151. Throughout this test, **show your work.**

1. (10 points) What is the exact value of $|8 - 2\pi| + |2\pi - 7|$. A solution that fails to show your understanding of the definition of absolute value is worth at most 1 point.

   Solution: First, note that $|8 - 2\pi| = 8 - 2\pi$. Then $|2\pi - 7| = 7 - 2\pi$. So the sum is $15 - 4\pi$.

2. (12 points)

   (a) For what value of k does the line $2y + kx = 6$ go through the point $(1, 4)$?
   Solution: Replacing x by 1 and y by 4, we have $2 \cdot 4 + k \cdot 1 = 6$, so $k = -2$.

   (b) Find the slope-intercept form of the line perpendicular to the line in (a) that includes the point $(1, 4)$.
   Solution: The slope of the line in (a) is 1, so the slope of the line we seek is -1. Therefore the line is $y - 4 = -1(x - 1)$, and the slope intercept form is $y = -x + 5$.

3. (15 points) Find the domain and range of each of the three functions below. Express your answers using interval notation. Use the letters D and R for domain and range, respectively.

   (a) $h(x) = \frac{x^2-1}{x-1}$.
   Solution: The only point that must be eliminated from the domain is $x = 1$ which causes division by zero. So the domain is $(-\infty, 1) \cup (1, \infty)$. The range is all real numbers except 2, $(-\infty, 2) \cup (2, \infty)$. Look at the graph and you'll see why.

   (b) $g(x) = \frac{\sqrt{2-x}}{x+4}$
   Solution: Here we must guard against the possibility that $2 - x$ is negative. So we solve $2 - x \geq 0$ to get $x \leq 2$ from which we have to remove the number $x = -4$. Thus, the domain is $(-\infty, -4) \cup (-4, 2]$. The second part, the range, is very difficult. The range is all the real numbers. Extra points to the first five students who come to my office to ask about this.

   (c) $f(x) = \begin{cases} x - 3 & \text{if } x < 2 \\ x + 2 & \text{if } x > 2 \end{cases}$
   Solution: Only 2 fails to belong to the domain: $(-\infty, 2) \cup (2, \infty)$. The range is the set $(-\infty, -1) \cup (4, \infty)$.

4. (52 points) Evaluate each of the limits indicated below.

(a) $\lim\limits_{x\to\infty} \dfrac{3x^4-6}{(11-3x^2)^3}$

Solution: The degree of the numerator is 4 while the degree of the denominator is 6 limit is 0.

(b) $\lim\limits_{x\to 1} \dfrac{x^4-1}{x^2-1}$

Solution: Factor both numerator and denominator to get $\lim_{x\to 1} \frac{x^4-1}{x^2-1} = \lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{(x^2-1)} = \lim_{x\to 1} \frac{x^2+1}{1} = 2$

(c) $\lim\limits_{x\to 1} \dfrac{(x-2)^3+1}{x-1}$.

Solution: Expand the numerator to get $\lim_{x\to 1} \frac{x^3-6x^2+12x-8+1}{x-1} = \lim_{x\to 1} \frac{x^3-6x^2+12x-7}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2-5x+7)}{x-1}$. So the zero over zero problem has disappeared. $= \lim\limits_{x\to 1} x^2-5x+7 = 3$.

(d) $\lim\limits_{x\to -2} \dfrac{x^2+2x}{x^2+5x+6}$

Solution: Factor and eliminate the $x+2$ from numerator and denominator to get

$$\lim_{x\to -2} \frac{x}{x+3} = -2$$

(e) $\lim\limits_{x\to 2} \dfrac{\frac{1}{4x}-\frac{1}{8}}{\frac{1}{2x}-\frac{1}{4}}$

Solution: The limit of both the numerator and the denominator is 0, so we must either factor or do the fractional arithmetic. Factoring seems to work best. The limit becomes

$$\lim_{x\to 2} \frac{\frac{1}{4}[\frac{1}{x}-\frac{1}{2}]}{\frac{1}{2}[\frac{1}{x}-\frac{1}{2}]} = \lim_{x\to 2} \frac{1}{4}\cdot\frac{2}{1} = \frac{1}{2}.$$

(f) $\lim\limits_{x\to 8} \dfrac{\sqrt{x+1}-3}{x-8}$

Solution: Rationalize the numerator to get

$$\lim_{x\to 8} \frac{x+1-9}{x-8}\frac{1}{\sqrt{x+1}+3} = \frac{1}{3+3} = \frac{1}{6}$$

For problems (g) through (m), let

$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2x^2 - 2 & \text{if } 0 \le x < 2 \\ 3 & \text{if } x = 2 \\ 10 - 3x & \text{if } x > 2 \end{cases}$$

(g) $\lim_{x \to 2^-} f(x)$
Solution: 6

(h) $\lim_{x \to 2^+} f(x)$
Solution: 4

(i) $\lim_{x \to 2} f(x)$
Solution: DNE

(j) $\lim_{x \to -1} f(x)$
Solution: -2

(k) $\lim_{x \to 3} f(x)$
Solution: 1

(l) $\lim_{x \to 0} f(x)$
Solution: -2

(m) Tell whether the function is continuous at each of the points:

i. $x = 0$
Solution: yes

ii. $x = 1$
Solution: yes

iii. $x = 2$
Solution: no

iv. $x = 3$
Solution: yes

5. (12 points) Let $H(x) = (x^2 - 1)(x + 2)^3$. Using the product rule,

$$H'(x) = (2x) \cdot (x + 2)^3 + 3(x^2 - 1) \cdot (x + 2)^2.$$

Find the three zeros of $H'(x)$.

Solution: Factor out the common terms to get $H'(x) = (x+2)^2[(2x)(x+2) + 3(x^2 - 1)]$. One zero is $x = -2$. The factor $[(2x)(x + 2) + 3(x^2 - 1)]$ simplifies to $5x^2 + 4x - 3$, which has two zeros. Apply the quadratic formula to get $x = \frac{-4 \pm \sqrt{16 - 4 \cdot 5(-3)}}{10}$ which reduces to $x = \frac{-4 \pm 2\sqrt{19}}{10} = \frac{-2 \pm \sqrt{19}}{5}$.

6. (12 points) Let $f(x) = 2x - \frac{1}{x}$ and let $g(x) = x^2 - 2$. Compute the composite functions listed below.

   (a) $f \circ g(x)$
   Solution: $f \circ g(x) = 2(x^2 - 2) - (x^2 - 2)^{-1}$.

   (b) $g \circ f(x)$
   Solution: $g \circ f(x) = (2x - 1/x)^2 - 2 = 4x^2 - 4 + 1/x^2 - 2 = 4x^2 - 6 + 1/x^2$.

   (c) $f \circ f(x)$
   Solution: $f \circ f(x) = 2(2x - 1/x) - \frac{1}{2x-1/x}$.

   (d) $g \circ g(x)$
   Solution: $g \circ g(x) = (x^2 - 2)^2 - 2 = x^4 - 4x^2 + 2$

7. (20 points) Let $f(x) = \frac{1}{x+1}$. Note that $f(0) = 1$.

   (a) Find the slope of the line joining the points $(0, 1)$ and $(0+h, f(0+h)) = (h, f(h))$, where $h \neq 0$. Then find the limit as h approaches 0 to get $f'(0)$.

   Solution: $\frac{f(h)-1}{h-0}$, which can be massaged to give $-\frac{1}{h+1}$. Thus $f'(0) = -1$.

   (b) Evaluate and simplify $\frac{f(x+h)-f(x)}{h}$. Then find the limit of the expression as h approaches 0.

   Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{(x+1)(x+h+1)}}{h} \\ &= -\frac{1}{(x+1)^2}. \end{aligned}$$

   (c) Replace the x with 0 in your answer to (b) to find $f'(0)$.

   Solution: $f'(0) = -1$

   (d) Use the information given and that found in (c) to find an equation in slope-intercept form for the line tangent to the graph of f at the point $(0, 1)$.

   Solution: The line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

8. (18 points) If a ball is thrown vertically upward from the roof of 128 foot building with a velocity of 64 ft/sec, its height after t seconds is $s(t) = 128 + 64t - 16t^2$.

   (a) What is the height the ball at time $t = 1$?

   Solution: $s(1) = 176$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) What is the maximum height the ball reaches?

   Solution: Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith. Thus, the max height is $s(2) = 128 + 64(2) - 16(2)^2 = 192$.

   (d) After how many seconds is the ball exactly 160 feet above the ground?

   Solution: Use the quadratic formula to solve $128 + 64t - 16t^2 = 160$. You get $t = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$.

   (e) How fast is the ball going the first time it reaches the height 160?

   Solution: Evaluate $s(t)$ when $t = 2 - \sqrt{2}$ to get $32\sqrt{2}$ feet per second.

   (f) How fast is the ball going the second time it reaches the height 160?

   Solution: Evaluate $s(t)$ when $t = 2 + \sqrt{2}$ to get $-32\sqrt{2}$ feet per second . In other words the ball is going downward at the same rate it was moving upwards when first went through 160 feet.