October 14, 2010 **Name** ____________________

The problems count as marked. The total number of points available is 172. Throughout this test, **show your work.**

1. (10 points) The points $(2, k)$ and $(5, 5)$ belong to the line perpendicular to the line $3x - 2y = 7$. Find the value of k.

   Solution: The given line has slope 3/2 so the one perpendicular has slope $-2/3$. Hence $\frac{k-5}{2-5} = -2/3$. Solving, we get $k = 7$.

2. (35 points) Evaluate each of the limits indicated below.

   (a) $\lim_{x\to\infty} \dfrac{3x^4 - 6}{(11 - 3x^2)^3}$

   Solution: The degrees of the numerator is 4 while the degree of the denominator is 6 limit is 0.

   (b) $\lim_{x\to 1} \dfrac{x^4 - 1}{x^2 - 1}$

   Solution: Factor both numerator and denominator to get $\lim_{x\to 1} \frac{x^4-1}{x^2-1} = \lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{(x^2-1)} = \lim_{x\to 1} \frac{x^2+1}{1} = 2$

   (c) $\lim_{h\to 0} \dfrac{(1+h)^3 - 1}{h}$.

   Solution: Expand the numerator to get

   $$\lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} = \lim_{h\to 0} \frac{3h + 3h^2 + h^3}{h} = \lim_{h\to 0} \frac{h(3 + 3h + h^2)}{h}$$

   $= \lim_{h\to 0}(3 + 3h + h^2)$, and now the zero over zero problem has disappeared. So the limit is 3.

   (d) $\lim_{x\to 1} \dfrac{x^2 - 4x + 3}{x^2 + x - 2}$

   Solution: Factor and eliminate the $x - 1$ from numerator and denominator to get

   $$\lim_{x\to 1} \frac{x - 3}{x + 2} = -2/3$$

(e) $\lim_{x\to 2} \frac{\frac{1}{3x} - \frac{1}{6}}{\frac{1}{2x} - \frac{1}{4}}$

Solution: The limit of both the numerator and the denominator is 0, so we must either factor or do the fractional arithmetic. Factoring seems to work best. The limit becomes

$$\lim_{x\to 2} \frac{\frac{1}{3}\left[\frac{1}{x} - \frac{1}{2}\right]}{\frac{1}{2}\left[\frac{1}{x} - \frac{1}{2}\right]} = \lim_{x\to 2} \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

(f) $\lim_{x\to -\infty} \frac{\sqrt{36x^2 - 3x}}{9x - 11}$

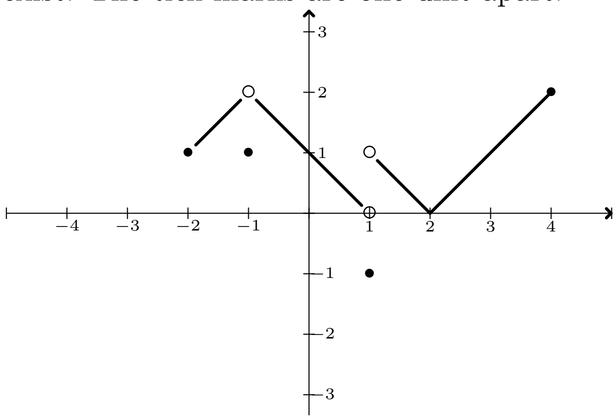
Solution: Divide both numerator and denominator by x to get $\lim_{x\to -\infty} \frac{-\sqrt{36-3/x}}{9-11/x} = 6/9 = -2/3$ because the degree of the denominator is essentially the same as that of the numerator.

(g) $\lim_{x\to 2} \frac{\sqrt{8x} - 4}{x - 2}$

Solution: Rationalize the numerator to get

$$\lim_{x\to 2} \frac{8x - 16}{x - 2} \frac{1}{\sqrt{8x} + 4} = 8 \cdot \frac{1}{8} = 1$$

3. (18 points) Consider the function F whose graph is given below. Evaluate each of the following expressions. Note: Enter 'DNE' if the limit does not exist. The tick marks are one unit apart.



(a) $\lim_{x \to -1^-} F(x) =$
Solution: 2

(b) $\lim_{x \to -1^+} F(x) =$
Solution: 2

(c) $\lim_{x \to -1} F(x) =$
Solution: 2

(d) $F(-1) =$
Solution: 1

(e) $\lim_{x \to 1^-} F(x) =$
Solution: 0

(f) $\lim_{x \to 1^+} F(x) =$
Solution: 1

(g) $\lim_{x \to 1} F(x) =$
Solution: dne

(h) $\lim_{x \to 3} F(x) =$
Solution: 1

(i) $F(3) =$
Solution: 1

4. (10 points) The points $(0, 0), (2, 1), (u, v)$, and $(1, -2)$ are the vertices of a square. Find u and v.

   Solution: One way to think about this is line segment through $(2, 1)$ and (u, v) must have the same slope and length as the segment from $(0, 0)$ to $(1, -2)$. But an easier way to think about this is that $(1, -2)$ is two down and one over from (0, 0, so (u, v) must be two down and one over from $(2, 1)$, That is $(u, v) = (3, -1)$.

5. (12 points) Find the domain of the function

$$g(x) = \frac{\sqrt{x+1}}{(x-1)(x-3)}.$$

Express your answer as a union of intervals. That is, use interval notation.

Solution: The numerator is defined for $x + 1 \geq 0$, that is $[-1, \infty)$. The denominator is zero at $x = 1$ and at $x = 3$, so these two numbers must be removed from $[-1, \infty)$. Thus, the domain is $[-1, 1) \cup (1, 3) \cup (3, \infty)$.

6. (12 points) Let $H(x) = (x^2 - 4)^2(2x + 3)^3$. Using the chain rule and the product rule,

$$H'(x) = 2(x^2 - 4) \cdot 2x(2x + 3)^3 + (x^2 - 4)^2 \cdot 3(2x + 3)^2 \cdot 2.$$

Three of the zeros of $H'(x)$ are $x = \pm 2$ and $x = -3/2$. Find the other two.

Solution: Factor out the common terms to get $H'(x) = (x^2-4)(2x+3)^2[4x+6(x^2-4)]$. One factor is $2x(2x+3)+3(x^2-4) = 7x^2+6x-12$. Apply the quadratic formula to get $x = \frac{-7\pm\sqrt{36-4\cdot 6(-12)}}{14}$ which reduces to $x = \frac{-7\pm 2\sqrt{93}}{14}$.

7. (40 points) Let $f(x) = 2x - \frac{1}{x}$ and let $g(x) = \sqrt{x^2+1}$. Compute, without simplifying, the composite functions listed below. Also use the product, quotient and chain rules to compute the derivatives listed.

   (a) $f \circ g(x)$
   Solution: $f \circ g(x) = 2\sqrt{x^2+1} - (x^2+1)^{-1/2}$.

   (b) $g \circ f(x)$
   Solution: $g \circ f(x) = \sqrt{(2x - 1/x)^2 + 1}$.

   (c) $f \circ f(x)$
   Solution: $f \circ f(x) = 2(2x - 1/x) - \frac{1}{2x - 1/x}$.

   (d) $g'(x)$
   Solution: $g'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$.

   (e) $f'(x)$
   Solution: By the sum and power rules, $f'(x) = 2 + x^{-2}$.

   (f) $\frac{d}{dx}[f \circ g(x)]$
   Solution: By the chain rule, $\frac{d}{dx} f \circ g(x) = [2 + ((x^2+1)^{1/2})^{-2}] \cdot \frac{x}{\sqrt{x^2+1}}$.

   (g) $\frac{d}{dx}[g \cdot f(x)]$
   Solution: $\frac{d}{dx}[g \cdot f(x)] = \frac{x}{\sqrt{x^2+1}} \cdot (2x - 1/x) + (2 + x^{-2})\sqrt{x^2+1}$.

   (h) $\frac{d}{dx}[f \div g(x)]$
   Solution: $\frac{d}{dx}[f \div g(x)] = \frac{(2+x^{-2})\sqrt{x^2+1} - \frac{x}{\sqrt{x^2+1}} \cdot (2x - x^{-1})}{x^2+1}$.

8. (10 points) Suppose $p(x)$ is a polynomial of degree 3 and $q(x)$ is a polynomial of degree 4. What is the degree of the polynomial $H(x) = (x^2p(x) - 1)^2 - (q(x) + x^2)^2 + x^8$? Write a sentence about your reasoning.

   Solution: We can reason as follows: $H(x)$ is the sum of three polynomials $(x^2p(x) - 1)^2, -(q(x) + x^2)^2$ and x^8, and their degrees are, respectively $10, 8$ and 8, so H has degree 10.

9. (15 points) Let

$$f(x) = \begin{cases} |x-3| & \text{if } x < 2 \\ s & \text{if } x = 2 \\ (t-x)^2 & \text{if } x > 2 \end{cases},$$

   where s and t are constants.

   (a) What is $\lim_{x\to 2^-} f(x)$?

   Solution: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} |x-3| = 1$.

   (b) For what values of t does $\lim_{x\to 2} f(x)$ exist.

   Solution: For the limit to exist, the limit from the right must be 1, so $t = 1$ and $t = 3$ both work.

   (c) If t is one of the values found in (b), for what value of s is f continuous at $x = 2$?

   Solution: For f to be continuous, $f(2)$ must be the limit as x approaches 2, which is 1. So $s = 1$ is the only number that works.

10. (10 points) The equation $x^2+4x+(y-1)^2 = 21$ is a circle. What is its radius?

    Solution: Complete the square to get $x^2 + 4x + 4 - 4 + (y-1)^2 = (x+2)^2 - 4 + (y-1)^2 = 21$. That is, $(x+2)^2 + (y-1)^2 = 25 = 5^2$. Hence the radius of the circle is 5.