October 15, 2008 **Name** ______________________________

The problems count as marked. The total number of points available is 133. Throughout this test, **show your work.**

1. (6 points) Find an equation (in slope-intercept form) for a line parallel to the line $3x - 6y = 7$ and which goes through the point $(-3, 5)$.

   Solution: The given line has slope 1/2 so the one parallel has slope 1/2 also. Hence $y - 5 = (1/2)(x + 3)$. Thus $y = x/2 + 13/2$.

2. (40 points) Evaluate each of the limits (and function values) indicated below.

   (a) $\lim_{x\to 3} \frac{x^2 + x - 12}{x^2 - 4x + 3}$

   Solution: Factor and eliminate the $x - 3$ from numerator and denominator to get

   $$\lim_{x\to 1} \frac{x+4}{x-1} = 7/2$$

   (b) $\lim_{x\to 3} \frac{x-3}{\frac{1}{x} - \frac{1}{3}}$

   Solution: The limit of both the numerator and the denominator is 0, so we must do the fractional arithmetic. The limit becomes

   $$\lim_{x\to 5} \frac{x-3}{\frac{3-x}{3x}} = \lim_{x\to 3} \frac{x-3}{-\frac{x-3}{3x}}$$
   $$= \lim_{x\to 3} \frac{1}{-\frac{1}{3x}} = -9.$$

   (c) $\lim_{x\to\infty} \frac{\sqrt{16x^2 - 3}}{11 - 5x}$

   Solution: Divide both numerator and denominator by x to get $\lim_{x\to\infty} \frac{\sqrt{16x^2/x^2 - 3/x^2}}{11/x - 5x/x} = -4/5$ because the degree of the denominator is essentially the same as that of the numerator.

   (d) $\lim_{x\to\infty} \frac{6x^5 - 3x^3}{11 - 12x^4}$

   Solution: The limit does not exist because the degree of the denominator is less than that of the numerator.

(e) $\lim_{x \to 1} \dfrac{x^2 - 1}{x^3 - 1}$

Solution: Factor both numerator and denominator to get $\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{x\to 1} \frac{(x-1)(x+1}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = 2/3$

(f) $\lim_{h \to 0} \dfrac{(2+h)^3 - 8}{h}$.

Solution: Expand the numerator to get $\lim_{h\to 0} \frac{8+12h+6h^2+h^3-8}{h} = \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} = \lim_{h\to 0} \frac{h(12+6h+h^2)}{h} = \lim_{h\to 0}(12 + 6h + h^2)$, and now the zero over zero problem has disappeared. So the limit is 12.

The following eight problems are worth 2 points each. For problems (g) through (n), let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x-1 & \text{if } 0 \le x < 2 \\ -1 & \text{if } x = 2 \\ 1 & \text{if } x > 2 \end{cases}$$

Find the value, if it exists, of each item below. Use DNE when the limit does not exist.

(g) $\lim_{x \to 0^-} f(x)$

Solution: 0

(h) $\lim_{x \to 0^+} f(x)$

Solution: -1

(i) $\lim_{x \to 0} f(x)$

Solution: DNE because the left limit and right limit are different.

(j) $f(0)$

Solution: -1

(k) $\lim_{x \to 2^-} f(x)$

Solution: 1

(l) $\lim_{x \to 2^+} f(x)$

Solution: 1, because both the left limit and the right limit are 1.

(m) $\lim_{x \to 2} f(x)$

Solution: 1

(n) $f(2)$

Solution: -1

3. (10 points) Find all the x-intercepts of the function

$$g(x) = 3(2x+7)^2(x-1)^2 - (2x+7)(x-1)^3.$$

Solution: Factor out the common terms to get $g(x) = (2x+7)(x-1)^2[3(2x+7-(x-1)] = (2x+7)(x-1)^2[5x+22]$. Setting each factor equal to zero, we find the zeros are $x = -7/2, x = 1$, and $x = -22/5$.

4. (15 points)

(a) Find all solutions of the inequality $\mid 2x-7 \mid \leq 5$ and write your solution in interval notation.

Solution: First solve the equation $\mid 2x-7 \mid \leq 5$, which has two solutions: $2x-7=5$ yields $x=6$ and $2x-7=-5$ yields $x=1$. Now consider the three intervals determined by these two points: $(-\infty,1),(1,6),(6,\infty)$. Select a test point from each of these intervals. I've picked $0, 3$, and 7. Trying each of these, we see that $\mid 2\cdot 0-7 \mid = 7 \leq 5$, NO; $\mid 2\cdot 3-7 \mid = 1 \leq 5$, YES; $\mid 2\cdot 7-7 \mid = 7 \leq 5$, NO; So only the interval $(1,6)$ works. Check the endpoints and see that they both work also. So our answer is $[1,6]$.

(b) Find the (implied) domain of

$$f(x) = \sqrt{\mid 2x-7 \mid -3},$$

and write your answer in interval notation.

Solution: We need to find out where $\mid 2x-7 \mid -3$ is zero or positive. First solve the equation $\mid 2x-7 \mid -3 = 0$ to get $x=5$ and $x=2$. Then use the test interval technique to find the sign chart for $g(x) = \mid 2x-7 \mid -3$. You see that $g(x) > 0$ on both $(-\infty,2)$ and $(5,\infty)$. Then notice that the endpoints need to be included. So the answer is $(-\infty,2] \cup [5,\infty)$.

5. (20 points) Let $f(x) = \frac{1}{x+1}$. Note that $f(0) = 1$.

   (a) Find the slope of the line joining the points $(0, 1)$ and $(0 + h, f(0 + h)) = (h, f(h))$, where $h \neq 0$. Then find the limit as h approaches 0 to get $f'(0$.

   Solution: $\frac{f(h)-1}{h-0}$, which can be massaged to give $-\frac{1}{h+1}$. Thus $f'(0) = -1$.

   (b) Evaluate and simplify $\frac{f(x+h)-f(x)}{h}$. Then find the limit of the expression as h approaches 0.

   Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\ &= \lim_{h \to 0} \frac{\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}}{h} \\ &= \lim_{h \to 0} \frac{\frac{-h}{(x+1)(x+h+1)}}{h} \\ &= -\frac{1}{(x+1)^2}. \end{aligned}$$

   (c) Replace the x with 0 in your answer to (b) to find $f'(0)$.

   Solution: $f'(0) = -1$

   (d) Use the information given and that found in (c) to find an equation for the line tangent to the graph of f at the point $(0, 1)$.

   Solution: The line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

6. (18 points) If a ball is thrown vertically upward from the roof of 128 foot building with a velocity of 64 ft/sec, its height after t seconds is $s(t) = 128 + 64t - 16t^2$.

   (a) What is the height the ball at time $t = 1$?

   Solution: $s(1) = 176$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) What is the maximum height the ball reaches?

   Solution: Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith. Thus, the max height is $s(2) = 128 + 64(2) - 16(2)^2 = 192$.

   (d) After how many seconds is the ball exactly 160 feet above the ground?

   Solution: Use the quadratic formula to solve $128 + 64t - 16t^2 = 160$. You get $t = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$.

   (e) How fast is the ball going the first time it reaches the height 160?

   Solution: Evaluate $s(t)$ when $t = 2 - \sqrt{2}$ to get $32\sqrt{2}$ feet per second.

   (f) How fast is the ball going the second time it reaches the height 160?

   Solution: Evaluate $s(t)$ when $t = 2 + \sqrt{2}$ to get $-32\sqrt{2}$ feet per second . In other words the ball is going downward at the same rate it was moving upwards when first went through 160 feet.

7. (24 points) Compute the following derivatives.

   (a) Let $f(x) = \frac{x^2-2x}{3x-x^2}$. Find $\frac{d}{dx}f(x)$.

   Solution: Use the quotient rule to get $f'(x) = \frac{d}{dx}f(x) = \frac{(2x-2)(3x-x^2)-(3-2x)(x^2-2x)}{(3x-x^2)^2}$. This can be simplified to $-\frac{x^2}{(3x-x^2)^2}$ which can be simplified still further: $-\frac{1}{(x-3)^2}$.

   (b) Let $g(x) = \sqrt{x^3+2x+4}$. What is $g'(x)$?

   Solution: $g'(x) = 1/2(x^3+2x+4)^{-1/2} \cdot (3x^2+2) = \frac{3x^2+2}{2\sqrt{x^3+2x+4}}$.

   (c) Find $\frac{d}{dx}((3x+1)^2 \cdot (4x^2-1))$
   Solution: Use the product rule to get $\frac{d}{dx}((3x+1)^2 \cdot (4x^2-1)) = 2(3x+1)^1 \cdot 3 \cdot (4x^2-1) + 8x(3x+1)^2$, which can be simplified but not significantly.

   (d) Let $f(x) = (2x^2+1)^4$. Find $f'(x)$.

Solution: Note that, by the chain rule, $f'(x) = 4(2x^2 + 1)^3 \cdot 4x = 16x(2x^2 + 1)^3$.