February 13, 2007 **Name** ______________________________

The problems count as marked. The total number of points available is 135. Throughout this test, **show your work.**

1. (40 points) Evaluate each of the limits indicated below.

   (a) $\lim_{x\to 0} \frac{x^4-x^2}{x^2}$
   Solution: Factor and eliminate the x^2 to get

   $$\lim_{x\to 1} x^2 - 1 = -1$$

   (b) $\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-1}$
   Solution: The limit of the numerator is 0 and the limit of the denominator is not zero, so $\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-1} = 0$.

   (c) $\lim_{x\to 5} \frac{x-5}{x^2-3x-10}$
   Solution: Factor and cancel $x-5$ to get

   $$\lim_{x\to 5} \frac{1}{x+2} = 1/7$$

   (d) $\lim_{x\to\infty} \frac{\sqrt{9x^2-3}}{11-5x}$

   Solution: $\lim_{x\to\infty} \frac{\sqrt{9x^2-3}}{11-5x} = \lim_{x\to\infty} \frac{\sqrt{9x^2/x^2-3/x^2}}{11/x-5x/x} = \frac{3}{-5} = -3/5$ because the degree of the denominator is essentially the same as that of the numerator.

   For problems (e) through (j), let

   $$f(x) = \begin{cases} 7-x & \text{if } x > 2 \\ 10 & \text{if } x = 2 \\ 2x+1 & \text{if } 0 \le x < 2 \\ -1 & \text{if } x < 0 \end{cases}$$

   (e) $\lim_{x\to 0^-} f(x)$
   Solution: -1

   (f) $\lim_{x\to 0^+} f(x)$
   Solution: 1

   (g) $\lim_{x\to 0} f(x)$
   Solution: DNE

(h) $\lim_{x\to 2^-} f(x)$
Solution: 5

(i) $\lim_{x\to 2^+} f(x)$
Solution: 5

(j) $\lim_{x\to 2} f(x)$
Solution: 5

2. (21 points) Consider the function whose properties are displayed.

a	-1	0	1	2	3	4
$\lim_{x\to a^-} f(x)$	DNE	1	1	4	2	3
$\lim_{x\to a^+} f(x)$	1	2	1	3	2	DNE
$f(a)$	1	2	-1	1	2	3
$\lim_{x\to a^-} g(x)$	4	1	3	3	1	0
$\lim_{x\to a^+} g(x)$	1	2	0	3	1	DNE
$g(a)$	1	-1	3	3	DNE	0

Using the table above calculate the limits below. Enter 'DNE' if the limit doesn't exist OR if limit can't be determined from the information given.

(a) $\lim_{x\to 2^+} [f(x) + g(x)]$
Solution: 6

(b) $\lim_{x\to 2^-} [f(x) + g(x)]$
Solution: 7

(c) $\lim_{x\to 2} [f(x) + g(x)]$
Solution: DNE

(d) $(f + g)(4)$
Solution: $3 + 0 = 3$.

(e) $f \circ g \circ f(-1)$
Solution: $f \circ g \circ f(-1) = f \circ g(1) = f(3) = 2$

(f) Find all points (in the table) at which f is continuous.
Solution: $x = 3$

(g) Find all points (in the table) at which g is continuous.
Solution: $x = 2$

3. (7 points) Compute the exact value of $|2-4\pi|+|8-2\pi|+|6-6\pi|$. No points for a decimal approximation.

   Solution: $|2-4\pi|+|8-2\pi|++|6-6\pi| = -(2-4\pi)+(8-2\pi)-(6-6\pi) = -2+4\pi+8-2\pi-6+6\pi = 8\pi$.

4. (10 points) Find the (implied) domain of

$$f(x) = \sqrt{(x-2)(x^2-9)},$$

   and write your answer in interval notation.

   Solution: Note that the domain must include those values of x for which the value inside the radical is at least zero. First factor $(x-2)(x^2-9)$ completely into $g(x) = (x-2)(x-3)(x+3)$. Then use the test interval technique to solve $(x-2)(x-3)(x+3) \geq 0$. You see that $g(x)$ is positive between -3 and 2 and to the right of 3. So the domain D of f is $D = [-3, 2] \cup [3, \infty)$.

5. (25 points) Let $f(x) = \sqrt{3x-2}$. Notice that $f(6) = \sqrt{18-2} = 4$.

   (a) Find the slope of the line joining the points $(6,4)$ and $(6+h, f(6+h))$, where $h \neq 0$. Note that $(6+h, f(6+h))$ is a point on the graph of f.
   Solution: $\frac{\sqrt{3(6+h)-2}-4}{6+h-6} = \frac{\sqrt{3(6+h)-2}-4}{h}$.

   (b) Compute $f(a+h)$, $f(a)$, and finally $\frac{f(a+h)-f(a)}{h}$.
   Solution:

   (c) Finally compute the limit as h approaches 0 to find $f'(a)$.

   Solution: Since we get zero over zero, we recall that, in this case, we should rationalize the numerator.

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{\sqrt{3(a+h)-2}-\sqrt{3a-2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(a+h)-2}-\sqrt{3a-2}}{h} \cdot \frac{\sqrt{3(a+h)-2}+\sqrt{3a-2}}{\sqrt{3(a+h)-2}+\sqrt{3a-2}} \\
&= \lim_{h\to 0} \frac{3(a+h)-2-(3a-2)}{h(\sqrt{3(a+h)-2}+\sqrt{3a-2})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(a+h)-2}+\sqrt{3a-2})} \\
&= \lim_{h\to 0} \frac{3}{(\sqrt{3(a+h)-2}+\sqrt{3a-2})} \\
&= \frac{3}{2(\sqrt{3a-2})}
\end{aligned}
$$

   (d) Replace the a with 6 to find $f'(6)$.
   Solution: $f'(6) = 3 \cdot 16^{-1/2}/2 = 3/8$

6. (32 points) Given three functions, $h(x) = 2x$,

$$g(x) = \begin{cases} 3x - 1 & \text{if } x > 6 \\ 4 - x & \text{if } x \leq 6 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} \sqrt{x+3} & \text{if } x \geq 1 \\ x^2 & \text{if } x < 1 \end{cases}$$

Note that $f \circ g \circ h(-2) = f \circ g(h(-2)) = f \circ g(-4) = f(8) = \sqrt{11}$.

(a) Complete the following table.

x	$h(x)$	$g \circ h(x)$	$f \circ g \circ h(x)$
-2	-4	8	$\sqrt{11}$
4			
	10		
		-2	
			0

Solution:

x	$h(x)$	$g \circ h(x)$	$f \circ g \circ h(x)$
-2	-4	8	$\sqrt{11}$
4	8	23	$\sqrt{26}$
5	10	29	$4\sqrt{2}$
3	6	-2	4
2	4	0	0

(b) Find all solutions to $f \circ g \circ h(x) = 3$.

Solution: $g(2x)$ could be 6 since $f(6) = \sqrt{9} = 3$. Also, $g(2x)$ could be $-\sqrt{3}$. These two equations lead to $x = -1$ and $4 - 2x = -\sqrt{3}$ from which it follows that $x = \sqrt{3}/2 + 2$.

(c) Find a symbolic representation of $g \circ h(x)$.

Solution:

$$g \circ h(x) = \begin{cases} 6x - 1 & \text{if } x > 3 \\ 4 - 2x & \text{if } x \leq 3 \end{cases}$$