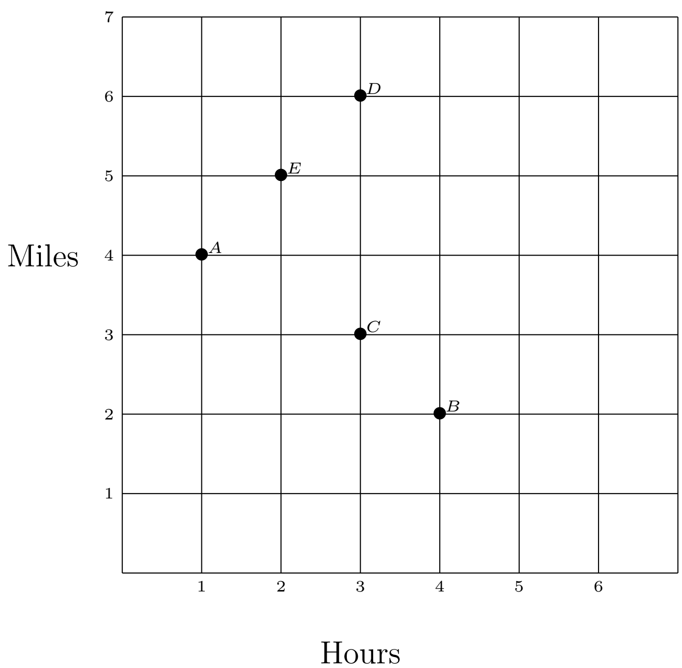
February 11, 2015 **Name** ______________________________

The problems count as marked. The total number of points available is 150. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit. Note please that this test is a composite of the tests for sections 1 and 2.

1. (6 points) Use the definition of absolute value to find the exact value of $|3\pi - 10 - \sqrt{2}| + |4 - \sqrt{2}|$. You might find it necessary to use the symbols π and/or $\sqrt{2}$.

   Solution: Using the definition of $|x|$, $|3\pi - 10 - \sqrt{2}| = 10 + \sqrt{2} - 3\pi$ and $|4 - \sqrt{2}| = 4 - \sqrt{2}$, so the sum is $14 - 3\pi$.

2. (10 points) Five hikers A, B, C, D and E recorded their distance hiked and time or various trails. List the hikers in order from slowest to fastest. Also, how much faster is the fastest hiker than the slowest hiker.



   Solution: The fastest is A who walks 4 miles per hour. The slowest is B at 2 miles per four hours, or half a mile per hour. So the difference is 3.5 miles per hour. In order, they are $B < C < D < E < A$. You can see that we are simply measuring the slopes of the lines.

3. (12 points) Let $A = (1, 2)$ and $B = (4, 6)$ be two points in the plane.

   (a) Find an equation for the line passing through both A and B.
   Solution: Using the point-slope form, $y - 2 = \frac{4}{3}(x - 1)$.

(b) Find an equation for the circle centered at A and passing through B.
Solution: The distance between A and B is $\sqrt{3^2+4^2}=5$, so the circle has the form $(x-1)^2+(y-2)^2=5^2$.

(c) Find the midpoint of the line segment joining A and B.
Solution: The midpoint is $(\frac{1+4}{2}, \frac{2+6}{2}) = (1.5, 4)$.

4. (59 points) Evaluate each of the limits (and function values) indicated below. It is very important to show your work on these problems. A correct 'naked' answer is worth 1 point.

(a) $\displaystyle\lim_{x\to 1} \frac{x^3+x^2+x-3}{x^3-3x^2+5x-3}$
Solution: Factor and eliminate the $x-1$ from numerator and denominator to get
$$\lim_{x\to 1} \frac{x^2+2x+3}{x^2-2x+3} = 6/2 = 3$$

(b) $\displaystyle\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$
Solution: The quotient rule for limits applies since the limit of the denominator function is not zero. So we have limit $=(1/2-1/3)(2-3) = -1/6$.

(c) $\displaystyle\lim_{x\to 7} \frac{\sqrt{x-3}-2}{x-7}$

Solution: Rationalize to get $\displaystyle\lim_{x\to 7} \frac{(\sqrt{x-3}-2)(\sqrt{x-3}+2)}{(x-7)(\sqrt{x-3}+2))} = \lim_{x\to 7} \frac{1}{\sqrt{x-3}+2} = \frac{1}{4}$.

(d) $\displaystyle\lim_{x\to -1} \frac{x^2-1}{x^3+1}$
Solution: Factor both numerator and denominator to get $\lim_{x\to -1} \frac{x^2-1}{x^3+1} = \lim_{x\to -1} \frac{(x+1)(x-1}{(x+1)(x^2-x+1)} = \lim_{x\to -1} \frac{x+1}{x^2-x+1} = 2$

(e) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^3-1}$
Solution: Factor both numerator and denominator to get $\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{x\to 1} \frac{(x-1)(x+1}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = 2/3$

(f) $\displaystyle\lim_{h\to 0} \frac{(2+h)^3-8}{h}$.

Solution: Expand the numerator to get $\lim_{h\to 0} \frac{8+12h+6h^2+h^3-8}{h} = \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} = \lim_{h\to 0} \frac{h(12+6h+h^2)}{h} = \lim_{h\to 0}(12+6h+h^2)$, and now the zero over zero problem has disappeared. So the limit is 12.

(g) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{16x^2-3}}{11-5x}$

Solution: Divide both numerator and denominator by x to get $\lim_{x\to\infty} \frac{\sqrt{16x^2/x^2-3/x^2}}{11/x-5x/x} = -4/5$ because the degree of the denominator is essentially the same as that of the numerator.

(h) $\displaystyle\lim_{x\to\infty} \frac{6x^5-3x^3}{11-12x^4}$

Solution: The limit does not exist because the degree of the denominator is less than that of the numerator.

(i) $\displaystyle\lim_{x\to\infty} \frac{6x^5-3x^3}{11-12x^5}$

Solution: The limit is simply $6/(-12) = -1/2$.

The following 10 problems are worth 1 point each. For problems below, let

$$f(x) = \begin{cases} 0 & \text{if } -3 < x < 0 \\ x-1 & \text{if } 0 \le x < 2 \\ -1 & \text{if } x = 2 \\ 1-x & \text{if } x > 2 \end{cases}$$

Find the value, if it exists, of each item below. Use DNE when the limit does not exist.

(j) What is the domain of the function f?

Solution:

(k) $\lim_{x\to 0^-} f(x)$

Solution: 0

(l) $\lim_{x\to 0^+} f(x)$

Solution: -1

(m) $\lim_{x\to 0} f(x)$

Solution: DNE because the left limit and right limit are different.

(n) $f(0)$

Solution: -1

(o) $\lim_{x\to 2^-} f(x)$

Solution: 1

(p) $\lim_{x\to 2^+} f(x)$

Solution: 1, because both the left limit and the right limit are 1.

(q) $\lim_{x\to 2} f(x)$

Solution: 1

(r) $f(2)$

Solution: -1

(s) Is f continuous at $x = 0$?

Solution: No, f does not even have a limit at 0.

5. (10 points) Find all the x-intercepts of the function

$$g(x) = 2(x-1)(2x+1)^2 + (x-1)^2(2x+1).$$

Solution: Factor out the common terms to get $g(x) = 2(x-1)(2x+1)[(2x+1) + 2(x-1)] = 2(x-1)(2x+1)(4x-1)$. Setting each factor equal to zero, we find the zeros are $x = 1, x - 1/2$ and $x = 1/4$.

6. (15 points)

(a) Find all solutions of the inequality $\mid 3x - 7 \mid \leq 5$ and write your solution in interval notation.

Solution: First solve the equation $\mid 3x-7 \mid = 5$, which has two solutions: $3x-7 = 5$ yields $x = 4$ and $3x-7 = -5$ yields $x = 2/3$. Now consider the three intervals determined by these two points: $(-\infty, 2/3), (2/3, 4), (4, \infty)$. Select a test point from each of these intervals. I've picked $0, 3,$ and 7. Trying each of these, we see that $\mid 3\cdot 0-7 \mid = 7 \leq 5$, NO; $\mid 3\cdot 3-7 \mid = 2 \leq 5$, YES; $\mid 3\cdot 7-7 \mid = 14 \leq 5$, NO; So only the interval $(2/3, 4)$ works. Check the endpoints and see that they both work also. So our answer is $[2/3, 4]$.

(b) Find the (implied) domain of

$$f(x) = \sqrt{\mid 3x-7 \mid -5},$$

and write your answer in interval notation.

Solution: We need to find out where $\mid 3x-7 \mid -5$ is zero or positive. First solve the equation $\mid 3x-7 \mid -5 = 0$ to get $x = 2/3$ and $x = 4$. Then use the test interval technique to find the sign chart for $g(x) = \mid 3x-7 \mid -5$. You see that $g(x) > 0$ on both $(-\infty, 2/3)$ and $(4, \infty)$. Then notice that the endpoints need to be included. So the answer is $(-\infty, 2/3] \cup [4, \infty)$.

7. (20 points) Let $f(x) = \sqrt{2x+1}$. Notice that $f(4) = \sqrt{2 \cdot 4 + 1} = 3$.

   (a) Find the slope of the line joining the two points $(4, f(4))$ and $(5, f(5))$.
   Solution: The slope is $\frac{f(5)-f(4)}{5-4} = \frac{\sqrt{11}-3}{1} \approx 0.317$.

   (b) Let h be a positive number. What is the slope of the line passing through the points $(4, f(4))$ and $(4+h, f(4+h))$. Your answer depends on h, of course.
   Solution: $\frac{f(4+h)-f(4)}{h} = \frac{\sqrt{2(4+h)+1}-\sqrt{2\cdot4+1}}{h}$.

   (c) Compute $\lim_{h\to0} \frac{f(4+h)-f(4)}{h}$ to get $f'(4)$.
   Solution: Since we get zero over zero, we recall that, in this case, we should rationalize the numerator.

$$
\begin{aligned}
\lim_{h\to0} \frac{f(4+h)-f(4)}{h} &= \lim_{h\to0} \frac{\sqrt{2(4+h)+1}-\sqrt{2\cdot4+1}}{h} \\
&= \lim_{h\to0} \frac{\sqrt{2(4+h)+1}-\sqrt{9}}{h} \cdot \frac{\sqrt{8+2h+1}+3}{\sqrt{8+2h+1}+3} \\
&= \lim_{h\to0} \frac{9+2h-9}{h(\sqrt{2(4+h)+1}+3)} \\
&= \lim_{h\to0} \frac{2h}{h(\sqrt{2(4+h)+1}+3)} \\
&= \lim_{h\to0} \frac{2}{(\sqrt{2(4+h)+1}+3)} \\
&= \frac{2}{2(3)} \\
&= \frac{1}{3}
\end{aligned}
$$

   So, $f'(4) = 1/3$.

   (d) Your answer to (c) is the slope of the line tangent to the graph of f at the point $(4, f(4))$. In other words, your answer is $f'(4)$. Write and equation for the tangent line.
   Solution: The line is $y - 3 = \frac{1}{3}(x-4)$, or $y = x/3 + 5/3$.

8. (20 points) Let $f(x) = \frac{1}{x+1}$. Note that $f(0) = 1$.

   (a) Find the slope of the line joining the points $(0, 1)$ and $(0+h, f(0+h)) = (h, f(h))$, where $h \neq 0$.

Solution: $\frac{f(h)-1}{h-0}$, which can be massaged to give $\frac{1}{h-1}$.

(b) Evaluate and simplify $\frac{f(x+h)-f(x)}{h}$. Then find the limit of the expression as h approaches 0.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{(x+1)(x+h+1)}}{h} \\ &= -\frac{1}{(x+1)^2}. \end{aligned}$$

(c) Replace the x with 0 in your answer to (b) to find $f'(0)$.

Solution: $f'(0) = -1$

(d) Use the information given and that found in (c) to find an equation for the line tangent to the graph of f at the point $(0, 1)$.

Solution: The line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

9. (12 points) Two circles C_1 and C_2 are given, $C_1 : x^2 + 4x + y^2 - 6y = 12$ and $C_2 : x^2 + y^2 - 2y = 0$.

   (a) What is the distance between the centers of the two circles.
   Solution: Put the circles in standard form by completing the square. $C_1 : x^2 + 4x + 4 + y^2 - 6y + 9 = 12 + 4 + 9 = 25$, so C_1 has center $(-2, 3)$ and radius 5, while circle C_2 has center $(0, 1)$ and radius 1. The distance between $(0, 1)$ and $(-2, 3)$ is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

   (b) Find an equation for the line joining the centers of the two circles.
   Solution: The line joining the centers is $y-1 = -1(x-0)$, or $y = -x+1$.

   (c) How many points belong to both circles?
   Solution: One circle lies entirely inside the other, so they have no points in common.

   (d) What is the distance from the point $P = (3, 5)$ to the point on C_1 that is closest to P?
   Solution: The point P is $\sqrt{29}$ from the center of C_1, so the distance from P to the circle is $\sqrt{29} - 5$.

10. (12 points) The midpoints of the segments AB joining $A = (1, 3)$ and $B = (-1, 7)$ and CD joining $C = (-2, 4)$ and $D = (4, 6)$ are joined by a line L.

    (a) What is the slope of the line L.
    Solution: The two midpoints are $(0, 5)$ and $(1, 5)$, so the slope of the line is $\frac{5-5}{1-0} = 0$.

    (b) How far apart are the two midpoints?
    Solution: The midpoints are 1 unit apart.

    (c) Find an equation for the line perpendicular to L and passing through the midpoint of the segment AB.
    Solution: Since L is horizontal, the perpendicular line is vertical. An equation for it is $x = 0$.

11. (12 points) Consider the parabola defined by $y = x^2 - 3x + 1$.

    (a) Write the equation in vertex form $y = a(x - h)^2 + k$ to find the vertex of the parabola.
    Solution: $y = x^2 - 3x + 1 = x^2 - 3x + 9/4 - 9/4 + 1 = (x - 3/2)^2 - 5/4$.

    (b) Use the information in (a) to find the smallest value of y among all the points on the parabola.
    Solution: Since $y = (x - 3/2)^2 - 5/4$, we can see that the least value of y occurs when $x = 3/2$, and that least value is $-5/4$.

12. (12 points) The vertices of a square are $(0, 1), (4, 4), (7, 0)$ and (u, v).

    (a) What is the area of the square?
    Solution: Each side has length $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$ so the area is 25.

    (b) What are the coordinates u and v?
    Solution: The point (u, v) must satisfy $u = 7-4 = 3$ and $v = 0-3 = -3$.