December 11, 2012 **Name** ______________________________

The total number of points available is 293. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (20 points) Derivative Problem.
   Let $f(x) = 2x^2 - x$. Then $f'(x) = 4x - 1$.

   (a) Use the limit definition of derivative to verify that $f'(x) = 4x - 1$.
   Solution:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{2(x+h)^2 - (x+h) - (2x^2 - x)}{h} \\
&= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) - x - h - 2x^2 + x}{h} \\
&= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - x - h - 2x^2 + x}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - h}{h} \\
&= \lim_{h\to 0} \frac{h(4x + 2h - 1)}{h} \\
&= \lim_{h\to 0} 4x + 2h - 1 \\
&= 4x - 1
\end{aligned}$$

   (b) Use the information above to find an interval over which $f(x)$ is increasing.
   Solution: Since $f'(x) > 0$ precisely when $x > 1/4$, $f(x)$ is increasing on $[1/4, \infty)$.

2. (30 points) Limit Problem

   (a) Find $\lim_{x\to 1} \frac{x^3+x^2+3x-5}{x^2-1}$.
   Solution: To resolve the zero over zero conflict, divide $x^3 + x^2 + 3x - 5$ by $x - 1$ to get $x^2 + 2x + 5$, then take the limit of the quotient obtained by eliminating the common factor $x - 1$. $\lim_{x\to 1} \frac{x^2+2x+5}{x+1} = 8/2 = 4$.

   (b) Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.

      i. Is it possible that $\lim_{x\to a} f(x) \cdot g(x) = 3$?
      Solution: The limit of the product is the product of the limits, so $\lim_{x\to a} f(x) \cdot g(x) = 0$.

      ii. Is it possible that $\lim_{x\to a} f(x)/g(x) = 3$?
      Solution: Yes, suppose $f(x) = 3x$ and $g(x) = x$. Then $\lim_{x\to a} f(x)/g(x) = 3$.

      iii. What are the possible outcomes of $\lim_{x\to a} \frac{f(x)}{g(x)}$? Can this limit fail to exist? Must the limit fail to exist? Write a sentence or two to show that you understand this question.
      Solution: This limit can fail to exist. It can also be any real number.

3. (15 points) Consider the function $f(x) = e^{x^3-3x^2-9x}$.

   (a) Find a value of x at which the line tangent to the graph of f is horizontal.
   Solution: Since $f'(x) = e^{x^3-3x^2-9x}(3x^2 - 6x - 9)$, we can factor and solve $3x^2 - 6x - 9 = 3(x + 1)(x - 3) = 0$ to get $x = -1$ and $x = 3$.

   (b) Find an interval over which $f(x)$ is increasing.
   Solution: Build the sign chart to see that f' is positive outside the interval $[-1, 3]$

   (c) Find an equation of the tangent line referred to in part a.
   Solution: $f(-1) = e^{-1-3+9} = e^5$, so the line is $y - e^5 = 0(x + 1)$, or $y = e^5$.

4. (30 points) There is a cubic polynomial $p(x)$ with zeros at $x = -2, x = 1$, and $x = 2$.

   (a) Build one such function.

   Solution: $f(x) = (x+2)(x-1)(x-2)$ is such a function.

   (b) Build the sign chart for your function.

   Solution: The function above is positive on $(-2, 1) \cup (2, \infty)$.

   (c) Find an interval over which your function is increasing?

   Solution: Build the sign chart for $f'(x) = 3x^2 - 2x - 4$. I'm getting $\frac{2 \pm 2\sqrt{13}}{6}$.

   (d) Find the area of the region bounded by your function over the interval from $x = -2$ to $x = 1$.

   Solution: First $f(x) = (x^2 - 4)(x - 1) = x^3 - x^2 - 4x + 4$, which is positive over $(-2, 1)$, so the area caught underneath the graph of f is $\int_{-2}^{1} x^3 - x^2 - 4x + 4dx = x^4/4 - x^3/3 - 2x^2 + 4x|_{-2}^{1} = 1/4 - 1/3 - 2 + 4 - (4 + 8/3 - 8 - 8) = 45/4$.

5. (12 points) Given $f''(x) = 2x - 6$ and $f'(-2) = 6$ and $f(-2) = 1$. Find $f'(x)$ and $f(x)$.

   Solution: First write $f'(x) = x^2 - 6x + C$ by the power rule. Solve $f'(-2) = 6$ for C to get $C = -10$. Then $f(x) = x^2 - 6x - 10$. Therefore $f(x) = \int x^2 - 6x - 10\ dx = x^3/3 - 3x^2 - 10x + C$. We can solve $f(-2) = 1$ to get $C = 19/3$, so the function is $f(x) = x^3/3 - 3x^2 - 10x + 19/3$.

6. (12 points) Let $f(x) = \dfrac{3}{x} - 2e^x$.

   (a) Find an antiderivative of $f(x)$.
   Solution: Note that $\int \frac{3}{x} - 2e^x \; dx = 3\ln x - 2e^x$.

   (b) Compute $\displaystyle\int_1^e f(x)dx$.

   Solution: Note that $\int \frac{3}{x} - 2e^x \; dx = 3\ln x - 2e^x$. So $\displaystyle\int_1^e f(x)dx = 3\ln x - 2e^x|_1^e = 3\ln e - 2e^e - (3\ln 1 - 2e) = 3 - 2e^e + 2e \approx -18.872$.

7. (42 points) Find the following antiderivatives.

   (a) $\displaystyle\int 2x - 5 \; dx$
   Solution: $x^2 - 5x + C$.

   (b) $\displaystyle\int 9x^2 - 4x - 2/x \; dx$
   Solution: $3 \cdot x^3 - 2 \cdot x^2 - 2\ln x + C$.

   (c) $\displaystyle\int \frac{3x^3 + 2x^2 - x}{x} \; dx$
   Solution: $\int (3x^3 + 2x^2 - x)/x \; dx = \int 3x^2 + 2x - 1 dx = x^3 + x^2 - x + C$.

   (d) $\displaystyle\int \frac{2x+3}{x^2 + 3x - 3} \; dx$
   Solution: By substitution, $(u = x^2 + 3x - 3)$, $\int \frac{2x+3}{x^2+3x-3} \; dx = \ln|x^2 + 3x - 3| + C$.

   (e) $\displaystyle\int 6x^5(x^6 + 3)^7 \; dx$

   Solution: By substitution with $u = x^6 + 3$, $\int 6x^4(x^6+3)^7 \; dx = \frac{(x^6+3)^8}{8} + C$.

   (f) $\displaystyle\int x^2 e^{x^3} \; dx$
   Solution: By substitution with $u = x^3, du = 3x^2$, $\int e^u \; du = e^u + C = (1/3)e^{x^3} + C$.

8. (15 points) Find the intervals over which $f(x) = x^2 e^{2x}$ is increasing.

   Solution: First $f'(x) = 2xe^{2x} + x^2 \cdot 2e^{2x} = 2e^{2x}(x + x^2)$, so we need to solve $x + x^2 = 0$ and we get $x = 0$ and $x = -1$. Since f' is negative precisely on $(-1, 0)$, f is increasing on $(-\infty, -1)$ and on $(0, \infty)$.

9. (12 points) Is there a value of b for which $\int_b^{2b} x^4 + x^2\, dx = 128/15$? If so, find it.

   Solution: Use the power rule to get the equation $(2b)^5/5 + (2b)^3/3 - (b^5/5 + b/3) = 128/15$. It follows that $b = 1$.

10. (30 points) Recall the $\ln(x)$ is defined for positive numbers only.

   (a) For which values of x is $g(x) = \ln\left(\frac{(x^2+1)(2x-1)}{(x^2-4)(3x+7)}\right)$ defined. Express your answer in interval notation. Notice that $g(0) = \ln\left(\frac{1(-1)}{(-4)7}\right) = \ln(1/28) = \ln(1) - \ln(28) = -\ln(28)$, so your answer above should include the number 0.

   Solution: Build the sign chart for $f(x) = \frac{(x^2+1)(2x-1)}{(x^2-4)(3x+7)}$. Note that there are four branch points, $x = 1/2$, $x = \pm 2$ and $x = -7/3$. So you see that $f(x) > 0$ precisely on $(-\infty, -7/3) \cup (-2, 1/2) \cup (2, \infty)$.

   (b) Find $g'(0)$.

   Solution: First, use the properties of $\ln(x)$ to simplify. $g(x) = \ln\left(\frac{(x^2+1)(2x-1)}{(x^2-4)(3x+7)}\right) = \ln(x^2+1) + \ln(2x-1) - \ln|(x^2-4)| - \ln(3x+7)$. Then $g'(x) = \frac{2x}{x^2+1} + \frac{2}{2x-1} - \frac{2x}{|x^2-4|} - \frac{3}{3x+7}$, whose value at zero is $g'(0) = -2 - 3/7 = -17/7$.

   (c) Use the information above to find an equation for the line tangent to $g(x)$ at the point $(0, -\ln(28))$.

   Solution: $y + \ln(28) = -(17/7)(x - 0)$ which is the same line as $y = -17x/7 - \ln(28)$.

11. (20 points) Use the substitution technique to find $\int(x-2)^4 \cdot x \; dx$. Then differentiate to check your answer.

    Solution: Let $u = x-2$, then $du = dx$ and $\int(x-2)^4 \cdot x \; dx = \int u^4(u+2) \; du$, and this give rise to $\frac{u^6}{6} + 2\frac{u^5}{5} + C = \frac{(x-2)^6}{6} + \frac{2(x-2)^5}{5} + C$.

12. (10 points) Find the derivative of the function g defined by $g(x) = \ln(e^{x^2-4x})$.

    Solution: Since $g(x) = \ln(e^{x^2-4x}) = x^2 - 4x$, it follows that $g'(x) = 2x - 4$.

13. (10 points) Compute $\int \frac{d}{dx}\ xe^{x^2}\ dx$.

    Solution: One function whose derivative is $\frac{d}{dx}\ xe^{x^2}$ is xe^{x^2}.

14. (15 points) Suppose x and y are positive real numbers satisfying $2xy = 9$.

    (a) Find two pairs of numbers (x_1, y_1) and (x_2, y_2) satisfying the condition $2xy = 9$. Compute the value of $2x + 3y$ for each of these pairs.

    Solution: Two possible points are $(1, 9/2)$ and $(2, 9/4)$, where the values are 15.5 and 10.75 respectively.

    (b) What is the smallest possible value of $2x + 3y$ such that $2xy = 9$.

    Solution: Minimize the function $f(x) = 2x + 27/2x$ to get $x = 3\sqrt{3}/2$.

    (c) What is the smallest possible value of $3x + 4y$ such that $2xy = 9$.

    Solution: This is similar to the one above.

15. (20 points) Use calculus to find the area of the trapezoid R bounded above by the graph of $f(x) = 2x + 1$, below by the x-axis, and on the sides by $x = 1$ and $x = 5$.

    Solution: $\int_1^5 2x + 1\ dx = x^2 + x|_1^5 = 5^2 + 5 - (1^2 + 1) = 30 - 2 = 28.$