December 14, 2009 **Name** ______________________________

The total number of points available is 282. Throughout this test, **show your work.** Throughout this test, the symbols DNE will mean 'does not exist'. In each of the first problems, circle the option that is closest to the correct answer. Each of the first 19 problems is worth 10 points.

1. Let $f(x) = x^4 - 5x + 4$. What is $f'(1)$?

   (A) -1 **(B)** 1 **(C)** 3 **(D)** 5 **(E)** 7

   Solution: A. $f'(x) = 4x^3 - 5$, so $f'(1) = 4 - 5 = -1$.

2. Which of the following intervals contains the number $|2\pi - 3\sqrt{5}| + |2\pi + \sqrt{5} - 8|$?

   (A) $[0, 0.2)$ **(B)** $[0.2, 0.4)$ **(C)** $[0.4, 0.6)$ **(D)** $[0.6, 0.8)$ **(E)** $[0.8, 1]$

   Solution: E. A calculator calculation shows that the answer is very close to 0.94.

3. What is the slope of a line perpendicular to the line $3x + 2y = 7$?

   (A) $2/3$ **(B)** $3/2$ **(C)** $-2/3$ **(D)** $-3/2$ **(E)** None of the above

   Solution: A. The slope of the given line is $-3/2$ so the slope we seek is $2/3$.

4. Which of the following belongs to the domain of $f(x) = \ln((x^2 + x - 2)(x^2 + 2x - 15))$?

   (A) -4 **(B)** -2 **(C)** -1 **(D)** 1 **(E)** 2

   Solution: Recall that $\ln(x)$ is defined if and only if $x > 0$. So build the sign chart for $u(x) = (x^2 + x - 2)(x^2 + 2x - 15) = (x-1)(x+2)(x+5)(x-3)$ to see that the only value of x that makes $u(x)$ positive is $x = -1$.

5. Suppose the line $3x + 2y = 7$ is tangent to the graph of $h(x)$ at the point $(1, 2)$. What is $h'(1)$?

   (A) $-3/2$ **(B)** $-2/3$ **(C)** 0 **(D)** $3/2$ **(E)** 7

   Solution: A. The slope of the line is $m = -3/2$.

6. What is $\displaystyle\lim_{x\to\infty} \frac{(6x-2)(2x-3)}{(3x+2)(4x-1)(x-1)}$?

   (A) 0 **(B)** $1/3$ **(C)** $1/2$ **(D)** $1/6$ **(E)** 1

   Solution: A. Using the asymptote theorem, since the degree of the denominator is larger, the limit is 0.

7. What is $\lim_{x \to -2} \dfrac{x^3+8}{x^2-4}$?

   (A) -3 **(B)** -2 **(C)** 2 **(D)** 3 **(E)** DNE

   Solution: D. Factor both numerator and denominator to get $\lim_{x\to -2} \frac{x^3+8}{x^2-4} = \lim_{x\to -2} \frac{(x+2)(x^2-2x+4)}{(x-2)(x+2)} = 12/-4 = -3$

8. Let $F(x)$ be an antiderivative of $3x^2 - 2x$. What is the growth of $F(x)$ over the interval $[1,5]$?

   (A) 18 **(B)** 27 **(C)** 36 **(D)** 100

   (E) The answer depends on which antiderivative is selected.

   Solution: D. One antiderivative is $F(x) = x^3 - x^2$ which grows from 0 to 100 on the given interval.

9. Let $H(x) = \ln(4x^2 + 12x + 10) - 2x$. One critical point of $H(x)$ is $x = -1$. Find another critical point.

   (A) $x = 0$ **(B)** $x = 2$ **(C)** $x = e$ **(D)** $x = 4$

   (E) There are no other critical points.

   Solution: E. We need to solve the equation $\frac{8x+12}{4x^2+12x+10} = 2$. This is equivalent to $8x^2 + 16x + 8 = 0$ which has repeated roots, $x = -1$

10. Let $g'(x) = (x-4)(x-2)(x+3)$. Over which one of the following intervals is g is increasing?

    (A) $[-6,-3]$ **(B)** $[-2,1]$ **(C)** $[0,3]$ **(D)** $[1,4]$ **(E)** $[3,6]$

    Solution: B. $g'(x) > 0$ on $(-3,2)$, so $[-2,1]$ is one of the intervals over which g is increasing. But $g'(x) < 0$ at some points of each of the others.

11. Which of the following is closest to the time required for a 12% investment to triple in value?

    (A) 5 years **(B)** 7 years **(C)** 9 years **(D)** 11 years **(E)** 13 years

    Solution: C. The method of compounding does not make much difference here. The triple time for continuous compounding is 9.15 years, and slightly more for others. For annual compounding, we get 9.69 years.

12. The half-life of a radioactive material is 100 years. How long does it take the material to lose one-third of its radioactivity?

    (A) 32 years **(B)** 40 years **(C)** 50 years **(D)** 58 years **(E)** 62 years

    Solution: It takes 58.5 years to lose down to 2/3 of its radioactivity.

13. What is the value of $\int_2^4 \frac{d(2x-5)^2}{dx}\, dx$

    (A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 10

    Solution: D. Its just $(2x-5)^2|_2^4 = 3^2 - (-1)^2 = 9 - 1 = 8$.

14. What is the area of the region R bounded above by $y = 2x - 3$, below by $y = x - 7$, on the left by $x = 2$ and on the right by $x = 6$?

    (A) 20 **(B)** 24 **(C)** 28 **(D)** 32 **(E)** 36

    Solution: D. Let $f(x) = 2x-3-(x-7) = x+4$. Now $\int_2^4 x+4 = x^2/2+4x|_2^6 = 18 + 24 - (2 + 8) = 32$.

15. Find a value of b for which $\displaystyle\int_b^{2b} x^2\, dx = 63$.

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

    Solution: B. Solve $\frac{x^3}{3}|_b^{2b} = 7b^2/3 = 63$ for b to get $b^3 = 27$ and $b = 3$.

16. The absolute maximum value of the function $f(x) = 2x^3 - 9x^2 + 12x + 14$ on the interval $-2 \leq x \leq 3$ is

    (A) 0 **(B)** 10 **(C)** 19 **(D)** 22 **(E)** 23

    Solution: E. Find $f'(x)$ first and then the critical points that are between -2 and 3. $f'(x) = 6x^2-18x+12 = 6(x^2-3x+2) = 6(x-1)(x-2)$, so there are two critical points and two endpoints to check. $f(-2) = -62$; $f(1) = 19$; $f(2) = 18$; and $f(3) = 23$, so the absolute maximum is $f(3) = 23$.

17. Two of the zeros of the polynomial $p(x) = (x-1)^3(x+2)^2 - 4(x-1)^2(x+2)$ are $x = 1$ and $x = -2$. There are two others. What is their sum?

    (A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

    Solution: B. Factor p to get $p(x) = (x-1)^2(x+2)[(x-1)(x+2) - 4]$. The sum of the two zeros of $(x-1)(x+2) - 4 = x^2 + x - 6 = (x+3)(x-2)$ is -1.

18. Rachel learns typing in a 14 week class. The number of words per minute Rachel can type after t weeks is given by

    $$F(t) = 160 - 40e^{-.4t}$$

    During which week does Rachel attain a speed of at least 155 words per minute.

    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

    Solution: D. Solve $F(t) = 155 = 160 - 40e^{-0.4t}$, so $t = \ln(1/8) \div -0.4$. Alternatively, Rachel's speed at the end of 5 weeks is just below 155, so it is during the sixth week that her speed goes over 155.

19. Consider the function $f(x) = xe^{2x}$. What is the slope of line tangent to the graph of f at the point $(\ln(3), 9\ln(3))$?

    (A) $3 + 6\ln(3)$ **(B)** $6\ln(3)$ **(C)** $9 + 18\ln(3)$

    (D) $9(1 + \ln(3))$ **(E)** $18(1 + \ln(3))$

    Solution: C. Since $f'(x) = e^{2x} + 2xe^{2x}$ by the product rule, $f'(\ln(3)) = 9 + 2\ln(3) \cdot 9$.

20. (30 points) Let $h(x) = \frac{x(2x-3)(2x+7)}{(x-1)^2(3x+12)}$.

(a) Find the zeros of h.

Solution: Solve $2x+7=0$ to get $x=-7/2$ and solve $2x-3=0$ to get $x=3/2$ for zeros, so the zeros are $x=3/2, x=-7/2$, and $x=0$.

(b) Find the asymptotes of h.

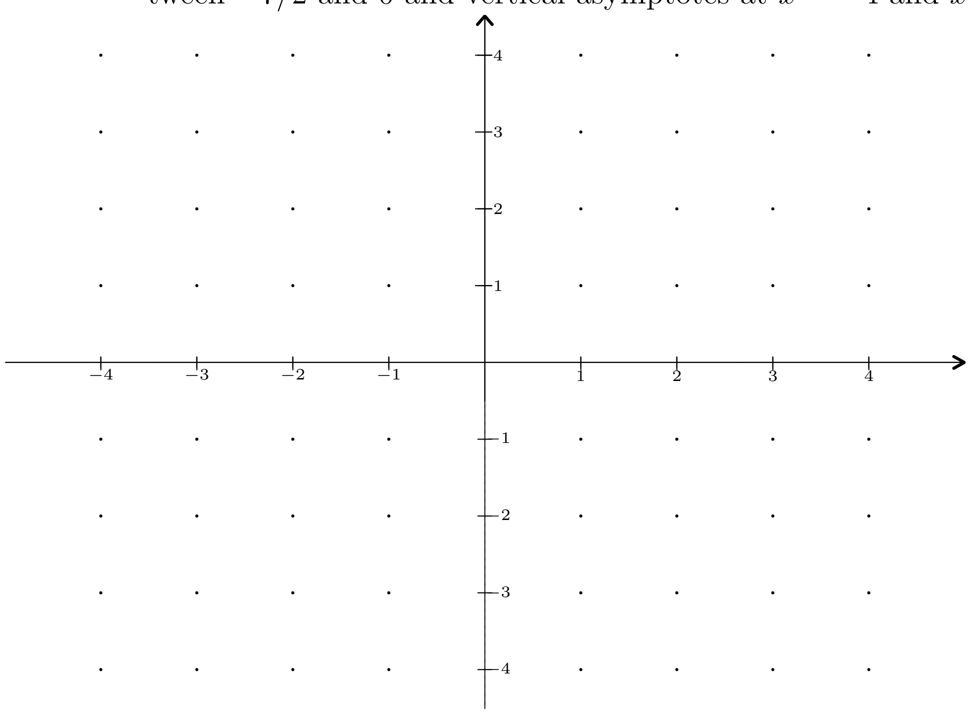
Solution: Solve $x-1=0$ and $3x+12=0$ to get $x=1$ and $x=-4$ as vertical asymptotes and $y=4/3$ as a horizontal asymptote.

(c) Build the sign chart for $h(x)$.

Solution: The sign chart shows that h is negative over $(-4,-7/2),(0,1)$ and $(1,3/2)$, and positive over $(3/2,\infty)$, $(-\infty,-4)$, $(-7/2,0)$.

(d) Sketch the graph of $h(x)$ USING the information in (a) and (b).

Solution: The graph must show that there is a relative maximum between $-7/2$ and 0 and vertical asymptotes at $x=-4$ and $x=1$.



21. (20 points) Let $H(x) = \sqrt{(3x+1)^9 + 2}$.

    (a) Find three functions f, g and h satisfying $f(g(h(x))) = f \circ g \circ h(x) = H(x)$.
    Solution: One solution is $h(x) = 3x + 1$, $g(x) = x^9 + 2$ and $f(x) = \sqrt{x}$.

    (b) Compute the derivative of each of the three component functions f, g, h.
    Solution: Using the three functions above, we have $h'(x) = 3$, $g'(x) = 9x^8$ and $f'(x) = x^{-1/2}/2$.

    (c) Apply the chain rule twice to find $H'(x)$.
    Solution: $H'(x) = \frac{1}{2}\left((3x+1)^9 + 2\right)^{-1/2} \cdot 9 \cdot (3x+1)^8 \cdot 3$.

22. (42 points) Find the following antiderivatives and definite integrals.

(a) $\displaystyle\int 3x - 5\ dx$

Solution: $3x^2/2 - 5x + C$.

(b) $\displaystyle\int_1^5 3x^2 - 2x + 7\ dx$

Solution: $x^3 - x^2 + 7x|_1^5 = 5^3 - 5^2 + 35 - (1^3 - 1^2 - 7) = 128$.

(c) $\displaystyle\int_0^1 \frac{d}{dx}e^{x^2}\ dx$

Solution: $e^{x^2}|_0^1 = e - 1$.

(d) $\displaystyle\int_e^3 \frac{1}{x}\ dx$

Solution: An antiderivative is $\ln|x|$. Therefore $\displaystyle\int_e^3 \frac{1}{x}\ dx = \ln|x|_e^3 = \ln 3 - \ln e = \ln 3 - 1 \approx 0.098$

(e) $\displaystyle\int_0^5 \frac{2x}{x^2+1}\ dx$

Solution: By substitution with $u = x^2+1$, $\int_0^5 \frac{2x}{x^2+1}\ dx$ becomes $\int 1/u du = \ln u + C$. Thus we get $\ln(x^2+1)|_1^5 = \ln 26 - \ln 1 = \ln 26 \approx 3.258$.

(f) $\displaystyle\int_0^1 2x(x^2+1)^4\ dx$

Solution: By substitution with $u = x^2 + 1, du = 2x\ dx$, $\int u^4\ du = u^5/5 = (x^2+1)^5/5|_0^1 = (2^5 - 1^5)/5 = 31/5$.