May 7, 2009 **Name** ______________________________

The total number of points available is 300. Throughout the free response part of this test, **show your work.** Throughout this test, the symbols DNE will mean 'does not exist' and $NOTA$ means 'none of the above'. In each of the first problems, circle the option that is closest to the correct answer. Each of the first 20 problems is worth 10 points.

1. Let $f(x) = x^3 - 2x + 4$. What is $f'(1)$?

   (A) 0 **(B)** 1 **(C)** 4 **(D)** 5 **(E)** $NOTA$

   Solution: B. $f'(x) = 3x^2 - 2$, so $f'(1) = 3 - 2 = 1$.

2. What is the y-intercept of the line tangent to the graph of $y = 3x^3 - 2x + 4$ at the point $(1, f(1))$?

   (A) 0 **(B)** -1 **(C)** -2 **(D)** -5 **(E)** $NOTA$

   Solution: C. Since $f'(x) = 9x^2 - 2$, $f'(1) = 7$, so the tangent line is $y - 5 = 7(x - 1)$, which has a y-intercept of -2.

3. Consider the function $f(x) = xe^{2x}$. What is the slope of line tangent to the graph of f at the point $(\ln(3), 9\ln(3))$?

   (A) $3 + 6\ln(3)$ **(B)** $6\ln(3)$ **(C)** $9 + 18\ln(3)$

   (D) $9(1 + \ln(3))$ **(E)** $18(1 + \ln(3))$

   Solution: C. Since $f'(x) = e^{2x} + 2xe^{2x}$ by the product rule, $f'(\ln(3)) = 9 + 2\ln(3) \cdot 9$.

4. Suppose the line $3x + 2y = 7$ is tangent to the graph of $h(x)$ at the point $(1, 2)$. What is $h'(1)$?

   (A) $-3/2$ **(B)** $-2/3$ **(C)** 0 **(D)** $3/2$ **(E)** 7

   Solution: A. The slope of the line is $m = -3/2$.

5. What is $\lim_{x\to\infty} \dfrac{(x-2)(2x-3)}{(3x+2)(4x-1)}$?

   (A) 0 **(B)** 1/3 **(C)** 1/2 **(D)** 1/6 **(E)** DNE

   Solution: D. Using the asymptote theorem, $\lim_{x\to\infty} \dfrac{(x-2)(2x-3)}{(3x+2)(4x-1)} = 2/12 = 1/6$.

6. What is the value of $|2\pi - 7| + |1-\pi| + \pi$?

   (A) 0 **(B)** $4\pi - 6$ **(C)** 6 **(D)** $6 - 2\pi$ **(E)** $8 + 4\pi$

   Solution: C. The value is $7 - 2\pi + \pi - 1 + \pi = 6$.

7. What is $\lim_{x\to 2} \dfrac{x^3-8}{x^2-4}$?

   (A) -3 **(B)** -2 **(C)** 2 **(D)** 3 **(E)** DNE

   Solution: D. Factor both numerator and denominator to get $\lim_{x\to 2} \frac{x^3-8}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = 12/4 = 3$

8. What is $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x^2-16}$?

   (A) 1/4 **(B)** 1/16 **(C)** 1/32 **(D)** 1/64 **(E)** DNE

   Solution: C. Rationalize the numerator and also factor the denominator to see that $x-4$ is a factor of both numerator and denominator. Then take limit to get 1/32.

9. Let $F(x)$ be an antiderivative of $3x^2 - 2x$. What is the growth of $F(x)$ over the interval $[1, 5]$?

   (A) 18 **(B)** 27 **(C)** 36 **(D)** 100

   (E) The answer depends on which antiderivative is selected.

   Solution: D. One antiderivative is $F(x) = x^3 - x^2$ which grows from 0 to 100 on the given interval.

10. Recall that one definition of $f'(x)$ is $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Use this idea to find $\lim_{h\to 0} \frac{(2+h)^6-64}{h}$.

    (A) 32 **(B)** 64 **(C)** 128 **(D)** 192 **(E)** DNE

    Solution: D. The derivative of x^6 is $6x^5$ and the value of this at $x = 2$ is $6 \cdot 2^5 = 192$.

11. Let $H(x) = \ln(4x^2 + 12x + 10) - 2x$. One critical point of $H(x)$ is $x = -1$. Find another critical point.

    (A) $x = 0$ **(B)** $x = 2$ **(C)** $x = e$ **(D)** $x = 4$

    (E) There are no other critical points.

    Solution: E. We need to solve the equation $\frac{8x+12}{4x^2+12x+10} = 2$. This is equivalent to $8x^2 + 16x + 8 = 0$ which has repeated roots, $x = -1$

12. Let $g(x) = 2x^3 - 7x^2 + 4x - 10$. Over which one of the following intervals is g is increasing?

    (A) $[-2, 1]$ **(B)** $[-1, 2]$ **(C)** $[0, 3]$ **(D)** $[1, 4]$ **(E)** $[2, 5]$

    Solution: E. $g'(x) < 0$ on $(1/3, 2)$, so $[2, 5]$ is the only one of the intervals over which g is increasing.

13. Let $k(x) = 2x^4 - 14x^3 + 30x^2 + 10x$. Over which of the following intervals is k is concave downwards?

    (A) $(0, 1)$ **(B)** $(1, 2)$ **(C)** $(2, 3)$ **(D)** $(3, 4)$ **(E)** $(4, 5)$

    Solution: B. $k''(x) = 12(2x - 5)(x - 1) < 0$ on $(1, 5/2)$.

14. What is the value of $\int_2^4 \frac{d(2x-5)^4}{dx}\ dx$

    (A) 20 **(B)** 40 **(C)** 60 **(D)** 80 **(E)** 100

    Solution: D. Since differentiation and antidifferentiation just undo each other, its just $(2x - 5)^4|_2^4 = 3^4 - (-1)^4 = 81 - 1 = 80$.

15. What is the area of the region R bounded above by $y = 2x - 3$, below by $y = x - 7$, on the left by $x = 2$ and on the right by $x = 4$?

    (A) 10 **(B)** 12 **(C)** 14 **(D)** 16 **(E)** 18

    Solution: C. Let $f(x) = 2x-3-(x-7) = x+4$. Now $\int_2^4 x+4 = x^2/2+4x|_2^4 = 8 + 16 - (2 + 8) = 14$.

16. Find a value of b for which $\displaystyle\int_b^{2b} \frac{1}{x} + 1\, dx = \ln(2) + 6$.

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

    Solution: C. Solve $\ln(2b) + 2b - \ln(b) - b = \ln(2) + b = \ln(2) + 6$ for b to get $b = 6$.

17. The absolute maximum value of the function $f(x) = 2x^3 - 9x^2 + 12x + 4$ on the interval $-2 \leq x \leq 3$ is

    (A) -10 **(B)** 0 **(C)** 9 **(D)** 12 **(E)** 13

    Solution: B. Find $f'(x)$ first and then the critical points that are between -2 and 3. $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$, so there are two critical points and two endpoints to check. $f(-2) = -72; f(1) = 9; f(2) = 8;$ and $f(3) = 13$, so the absolute maximum is $f(3) = 13$, which of course occurs at $x = 3$.

18. Two of the zeros of the polynomial $p(x) = (x-1)^3(x+2)^2 - 4(x-1)^2(x+2)$ are $x = 1$ and $x = -2$. There are two others. What is their sum?

    (A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

    Solution: B. Factor p to get $p(x) = (x-1)^2(x+2)[(x-1)(x+2) - 4]$. The sum of the two zeros of $(x-1)(x+2) - 4 = x^2 + x - 6 = (x+3)(x-2)$ is -1.

19. Rachel learns typing in a 14 week class. The number of words per minute Rachel can type after t weeks is given by

    $$F(t) = 120 - 40e^{-.4t}$$

    Which of the following is closest to the increase in the number of words per minute Rachel can type during the second week of the course?

    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

    Solution: E. $F'(t) = 16e^{-0.4t}$, so $F'(2) = 16e^{-0.8} \approx 7.18$ words per minute.

20. Given that the graph of f passes through the point $(1, 5)$ and that the slope of its tangent line at $(x, f(x))$ is $2x + 1$, what is $f(4)$?

    (A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 28

    Solution: D. $f(x) = x^2 + x + c$ so $f(1) = 1 + 1 + c = 5$. It follows that $c = 3$. Thus, $f(4) = 4^2 + 4 + 3 = 23$.

21. (30 points) Let $h(x) = \frac{x(2x+11)(2x+7)}{(x-1)^2(3x-12)}$.

    (a) Find the asymptotes and the zeros of h.

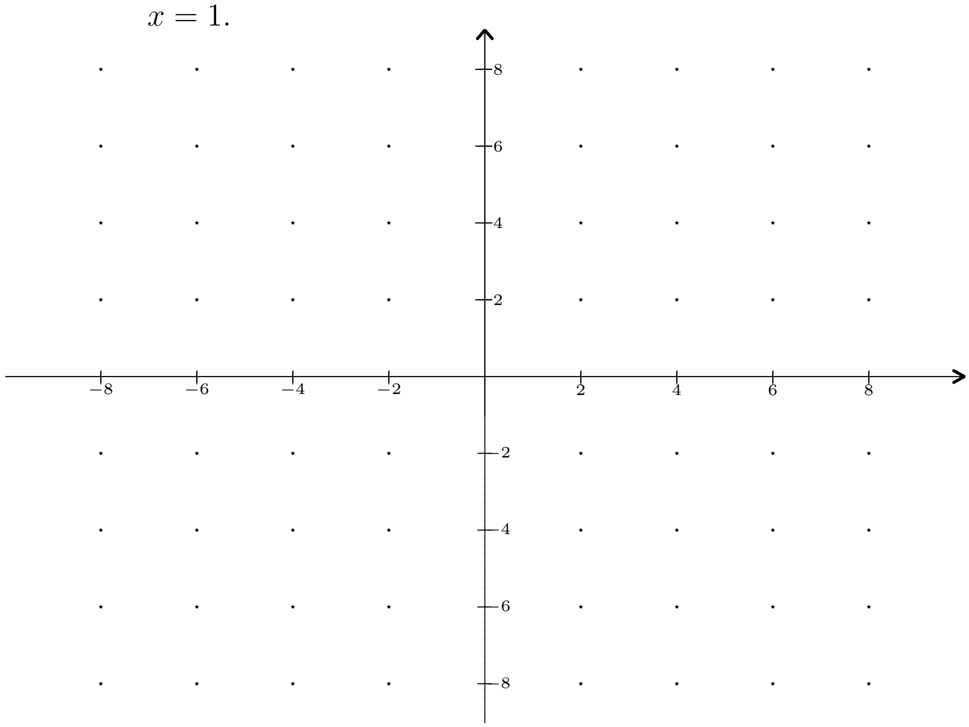
    Solution: Solve $2x + 7 = 0$ to get $x = -7/2$ and solve $2x + 11 = 0$ to get $x = -11/2$ for zeros along with $x = 0$, and $x = 1, x = 4$, and $y = 4/3$ for asymptotes.

    (b) Build the sign chart for $h(x)$.

    Solution: The sign chart shows that h is positive over $(-\infty, -11/2), (-7/2, 0)$ and $(4, \infty)$, and negative on the open intervals $(-11/2, -7/2)$, $(0, 1)$ and $(1, 4)$.

    (c) Sketch the graph of $h(x)$ USING the information in (a) and (b).

    Solution: The graph must show that there are relative extrema at two values, a minimum between -2 and -3 and a maximum between 1 and 2.5. Also, your graph must make clear that there is not sign change at $x = 1$.



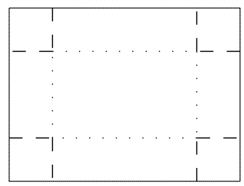
22. (20 points) Let $H(x) = \sqrt{(2x+1)^{10}+3}$.

   (a) Find three functions f, g and h satisfying $f(g(h(x))) = f \circ g \circ h(x) = H(x)$.
   Solution: One way to do this is to let $f(x) = \sqrt{x}$, $g(x) = x^{10} + 3$, and $h(x) = 2x + 1$.

   (b) Compute the derivative of each of the three component functions f, g, h.
   Solution: In case we choose the functions above, we get $f'(x) = x^{-1/2}/2$, $g'(x) = 10x^9$, and $h'(x) = 2$.

   (c) Apply the chain rule twice to find $H'(x)$.
   Solution: $H'(x) = f'(g(h(x)) \cdot g'(h(x)) \cdot h'(x) = \frac{1}{2}((2x+1)^{10}+3)^{-1/2} \cdot 10(2x+1)^9 \cdot 2 = 10((2x+1)^{10}+3)^{-1/2} \cdot (2x+1)^9$.

23. (20 points) For the problem below, let F be the number of letters in your first (= given) name, and let L be the number of letters in your last (= family) name. The point is to customize the problems for you. Four congruent $x \times x$ squares from the corners of a cardboard rectangle that measures $2F \times 2L$. The sides are then folded upward to form a topless box. Find the volume V as a function of x. What is the logical domain? Compute $V(0), V(1), V(2)$, and $V(3)$. Find $V'(x)$ and use this to determine the critical points of V. Find the absolute maximum value of V and the value of x where it occurs.



Solution: The following solution is meant to satisfy everyone. Your solution will not look like this because your F and L are determined by your name. However, in order to make the solution relevant to everyone, I'm using F and L throughout. First $V(0) = 0, V(1) = 1(2F-2)(2L-2), V(2) = 2(2F-4)(2L-4)$, and $V(3)+3(2F-6)(2L-6)$. More generally, $V(x) = x(2F-2x)(2L-2x) = 4x(F-x)(L-x)$. Therefore

$$\begin{aligned}
V'(x) &= 4(F-x)(L-x) + 4x(F-x)(-1) + 4x(L-x)(-1) \\
&= 4(F-x)(L-x) - 4x(F-x) - 4x(L-x) \\
&= 4[(F-x)(L-x) - x(F-x) - x(L-x)] \\
&= FL - Fx - Lx + x^2 - Fx + x^2 - Lx + x^2 = FL - 2Fx - 2Lx + 3x^2.
\end{aligned}$$

Thus we need to solve $3x^2 - (2F+2L)x + FL = 0$. By the quadratic formula,

$$\begin{aligned}
x &= \frac{2F + 2L \pm \sqrt{4F^2 + 8FL + 4L^2 - 4 \cdot 3FL}}{6} \\
&= \frac{2F + 2L \pm 2\sqrt{F^2 - FL + L^2}}{6} \\
&= \frac{2F + 2L \pm 2\sqrt{F^2 - FL + L^2}}{6} \\
&= \frac{F + L \pm \sqrt{F^2 - FL + L^2}}{3}
\end{aligned}$$

24. (30 points) The quadrilateral T with vertices $A = (0,0)$, $B = (0,6)$, $C = (8,10)$ and $D = (8,0)$ is a trapezoid since the two sides AB and CD are both vertical. It is not hard to see that the area of T is 64 square units.

    (a) Find an equation for the line passing through the points B and C. Let $f(x)$ be the function whose graph is this line.

    Solution: The slope of the line is $m = \frac{10-6}{8-0} = \frac{1}{2}$, so the line is $y - 6 = \frac{1}{2}(x - 0)$, which is $y = x/2 + 6$

    (b) Use calculus, showing all you work, to verify that the area of the region T bounded above by the graph of f, below by the x-axis, and on the sides by $x = 0$ and $x = 8$ is 64.

    Solution: Since the function is positive, the area is the same as the integral, $\int_0^8 x/2 + 6\ dx = x^2/4 + 6x|_0^8 = 64/4 + 6 \cdot 8 - 0 = 16 + 48 = 64$.