May 6, 2008 Name ____________

The total number of points available on this test is 239. **Show all your work.** If you use decimal notation, carry out the arithmetic to three places to the right of the decimal and round off to two places.

1. (12 points) The line tangent to the graph of a function f at the point $(2,5)$ on the graph also goes through the point $(0,11)$. What is $f'(2)$?

   Solution: The slope of the line through $(2,5)$ and $(0,11)$ is $\frac{11-5}{0-2} = -3$. In other words, $f'(3) = -3$.

2. (20 points) Let $f(x) = \sqrt{x-2}$.

   (a) Compute the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

   Solution: $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{x+h-2}-\sqrt{x-2}}{h}$.

   (b) Compute $lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

   Solution: $lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = lim_{h\to 0}\dfrac{\sqrt{x+h-2}-\sqrt{x-2}}{h}$. We cannot use the quotient rule for limits because that would give $\frac{0}{0}$, so we have to dig into our repertoire to find the right method. Its rationalizing the numerator! When we do this we get $\frac{x+h-2-(x-2)}{h(\sqrt{x+h-2}+\sqrt{x-2})}$. Take limit to get $\frac{1}{2\sqrt{x-2}}$.

   (c) Use the information in part (b) to find $f'(3)$.
   Solution: $f'(3) = \frac{1}{2\sqrt{3-2}} = \frac{1}{2}$.

   (d) Use the information above to find an equation for the line tangent to f at the point $(3, f(3))$.
   Solution: The slope is $m = \frac{1}{2}$ so the line is $y - 1 = \frac{1}{2}(x-3)$ which in slope-intercept form is $y = x/2 - 1/2$.

3. (12 points) Suppose $f'(x) = x - \ln(x)$ and $f(e) = 3$. Find an equation for the line tangent to the graph of f at the point $(e, 3)$.

   Solution: $y - 3 = (e-1)(x-e)$.

4. (15 points) Let $f(x) = \sqrt{9 - |x-5|}$. Use the test interval method to find the domain of f.

Solution: Solve $9-|x-5|=0$ to get the two values $x=-4$ and $x=14$. Then build the sign chart for $9-|x-5|$ to see that the solution to $9-|x-5|\geq 0$ is $[-4,14]$.

5. (12 points)

   (a) Find the rate of change of $f(x)=x^2\ln(2x+1)$ when $x=1$.
   Solution: Use the product rule to get $f'(x)=2x\ln(2x+1)+\frac{2}{2x+1}\cdot x^2$ whose value at $x=1$ is $f'(1)=2\ln(3)+(2/3)\approx 2.86$.

   (b) Find the slope of the line tangent to f is the point $(2,4\ln 5)$.
   Solution: $f'(2)=4\ln 5+\frac{8}{5}\approx 8.04$.

6. (12 points) A radioactive substance has a half-life of 37 years. Find an expression for the amount of the substance at time t if 20 grams were present initially.

   Solution: $Q(t)=Q_0e^{-kt}$. Since $Q_0=20$ and the half-life is 37 years, it follows that $10=20e^{-37k}$, which can be solved to give $k\approx 0.01873$. Thus $Q(t)=20e^{-0.018732t}$.

7. (12 points) If $h=g\circ f$ and $f(1)=2, g'(2)=5, f'(1)=-3$ find $h'(1)$.

   Solution: $h'(1)=g'(f(1))\cdot f'(1)=g'(2)\cdot f'(1)=-15$.

8. (15 points) Let $f(x)=e^{2x}/x^2$. Find the interval(s) where f is concave upward.

   Solution: Use the quotient rule to get $f'(x)=\frac{2e^{2x}\cdot x^2-2xe^{2x}}{x^4}=\frac{2e^{2x}(x-1)}{x^3}$ and $f''(x)=2e^{2x}\cdot\frac{x^3-3x^2(x-1)}{x^6}+\frac{4e^{2x}(x-1)}{x^3}$, which after massaging, gets to be $2e^{2x}\left(\frac{2x^2-4x+3}{x^4}\right)$, which has no zeros. Therefore the function is convex upwards each of the pieces of its domain, $(-\infty,0)$ and $(0,\infty)$.

9. (15 points) Find the area of the region R bounded above by the graph of $f(x)=-(x+1)(x-3)$, below by the x-axis, and on the sides by the vertical lines $x=0$ and $x=3$.

   Solution: The integral is $\int_0^3 -(x+1)(x-3)\ dx=-\int_0^2 x^2-2x-3\ dx=-\left(x^3/3-x^2-3x|_0^3\right)=9-9+9=9$.

10. (15 points) Find the area of the region R caught between the graph of $f(x)=x^2-3x+2$ and $g(x)=-x+5$.

    Solution: First note that the two functions agree at $x=-1$ and $x=3$. You can make this discovery either by factoring the difference function or using the graphing calculator. The area then is the value of the integral $\int_{-1}^{3} g(x) - f(x)\, dx = \int_{-1}^{3} -x^2 + 2x + 3\, dx = -x^3/3 + x^2 + 3x|_{-1}^{3} = 9 + 1/3 - 4 = 32/3$.

11. (15 points) Find all asymptotes of the rational function $r(x) = \dfrac{(x^3-64)(x^2-9)}{3(x^2-16)(x+3)(x^2)}$.

    Solution: Factor completely and remove like factors to get $r(x) = \dfrac{(x-4)(x^2+4x+16)(x-3)(x+3)}{3(x-4)(x+4)(x+3)(x^2)} = \dfrac{(x^2+4x+16)(x-3)}{3(x+4)(x^2)}$. At this point we can read off the asymptotes: $x = -4, x = 0$, and $y = 1/3$.

12. (18 points) If a ball is thrown vertically upward from the roof of 128 foot building with a velocity of 64 ft/sec, its height after t seconds is $s(t) = 128 + 64t - 16t^2$. Be sure to show your work and explain each step in English.

    (a) What is the height the ball at time $t = 1$?
    Solution: $s(1) = 176$.

    (b) What is the velocity of the ball at the time it reaches its maximum height?
    Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

    (c) What is the maximum height the ball reaches?
    Solution: Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith. Thus, the max height is $s(2) = 128 + 64(2) - 16(2)^2 = 192$.

    (d) After how many seconds is the ball exactly 160 feet above the ground?
    Solution: Use the quadratic formula to solve $128 + 64t - 16t^2 = 160$. You get $t = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$.

    (e) How fast is the ball going the first time it reaches the height 160?
    Solution: Evaluate $s(t)$ when $t = 2 - \sqrt{2}$ to get $32\sqrt{2} \approx 45.25$ feet per second.

    (f) How fast is the ball going the second time it reaches the height 160?
    Solution: Evaluate $s(t)$ when $t = 2 + \sqrt{2}$to get $-32\sqrt{2}$. In other words the ball is going downward at the same rate it was moving upwards when first went through 160 feet.

13. (10 points) Evaluate $\displaystyle\int x^2 - \sqrt{x} - \frac{1}{x}\, dx$

    Solution: $\int x^2 - \sqrt{x} - \frac{1}{x}\, dx = \int x^2\, dx - \int \sqrt{x}\, dx - \int -\frac{1}{x}\, dx = x^3/3 - 2x^{3/2}/3 + \ln|x| + C$.

14. (10 points) Evaluate $\int x^2\sqrt{x^3+4}\,dx$

    Solution: Use substitution with $u = x^3+4$ to get $1/3\int u^{1/2}\,du = 2u^{3/2}/9 = 2(x^3+4)^{3/2}/9 + C$.

15. (36 points) Evaluate each of the following integrals using the Fundamental Theorem of Calculus (ie, antidifferentiate, then measure the growth of an antiderivative over the interval).

    (a) Evaluate $\displaystyle\int_0^4 \frac{x^3+8}{x+2}\,dx$

    Solution: Factor the numerator to get $\displaystyle\int_0^4 \frac{x^3+8}{x+2}\,dx = \int_0^4 \frac{(x+2)(x^2-2x+4)}{x+2} = \int_0^4 x^2-2x+4 = x^3/3 - x^2 + 4x|_0^4 = 64/3 - 16 + 16 = 64/3.$

    (b) Evaluate $\displaystyle\int_1^3 x^3\cdot(x^4-2)^2\,dx$

    Solution: Let $u = x^4-2$. Then $du = 4x^3dx$ and $\int_1^3 x^3\cdot(x^4-2)^2\,dx = \frac{1}{4}(x^4-2)^3|_1^3 = \frac{79^3}{12} - \frac{-1}{12} = 41086.5.$

    (c) Evaluate $\displaystyle\int_0^4 2xe^{x^2}\,dx$

    Solution: $\int_0^4 2xe^{x^2}\,dx = e^{x^2}|_0^4 = e^{16} - e^0 \approx 8886110.5 - 1 = 8886109.5.$