May 6, 2019 **Name** ______________________________

The problems count as marked. The total number of points available is xxx. Throughout this test, **show your work.** Please note that a few problems that appear here were removed from the final because it was too long. Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (10 points) The function f satisfies $f(2) = 3$. The line tangent to f at $(2, 3)$ can be represented as $y - 3 = 5(x - 2)$, What is $f'(2)$. Explain your answer with a complete sentence.

   Solution: $f'(2) = 5$ because the slope of the tangent line is 5.

2. (15 points) Consider the cubic curve $f(x) = 2x^3 - 4.5x^2 - 27x + 2$.

   (a) Build the sign chart for $f'(x)$.
   Solution: $f'(x) = 6x^2 - 9x - 27 = 3(2x^2 - 3x - 9) = 3(2x - 3)(x + 3)$, which is negative over $(-3, 3/2)$ and positive elsewhere.

   (b) Using the sign chart for $f'(x)$, find the intervals over which $f(x)$ is increasing.

   Solution: Since $f'(x)$ is positive on $(-\infty, -3)$ and on $(3/2, \infty)$, f is increasing over those intervals.

   (c) Find a point of inflection on the graph of f.
   Solution: $f''(x) = 12x + -9 = 3(4x - 3)$, which changes signs at $x = 4/3$, so there is a point of inflection at $(4/3, f(4/3))$.

3. (36 points) Demonstrate your understanding of the product, quotient and chain rules by differentiating each of the given functions. Find the critical points for each function and the intervals over which the function is increasing. You must show your work.

   (a) Let $F(x) = (2x+8)(4x-6)$

   Solution: Note that $F'(x) = 2(4x-6) + 4(2x+8) = 16x + 20$, so the only critical point is $x = -20/16$. Since $f'(x) > 0$ on $(-20/16, \infty)$, we conclude that F is increasing on that interval.

   (b) $G(x) = \dfrac{x^2 - 3x + 15/2}{2x-1}$

   Solution: By the quotient rule, $G'(x) = \frac{(2x-3)(2x-1)-2(x^2-3x+15/2)}{(2x-1)^2} = \frac{2x^2-2x-12}{(2x-1)^2}$. So the critical points are $x = -2$ and $x = 3$. We can see that $G' > 0$ on both $(-\infty, -2)$ and on $(3, \infty)$. So G is increasing on these two intervals.

   (c) $K(x) = (x^2-4)^{18}$

   Solution: By the chain rule, $K'(x) = 18(x^2-4)^{17} \cdot 2x$, so the critical points are $x = -2, 0, 2$. Build the sign chart for K' to see that K' is positive on both $(-2, 0)$ and $(2, \infty)$. Therefore K is increasing on those two intervals.

4. (30 points) Build a rational function $r(x) = \frac{p(x)}{q(x)}$ satisfying

   (a) r has vertical asymptotes at $x = \pm 3$

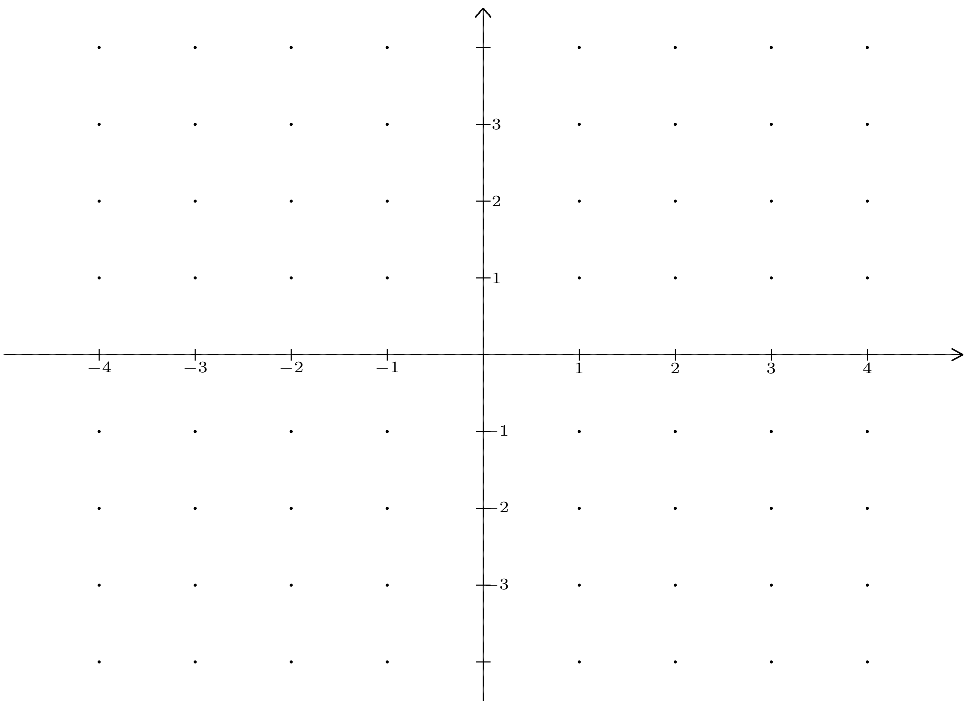
   (b) r has zeros at $x = 0$ and $x = 1$

   (c) r has the horizontal asymptote is $y = 2$.

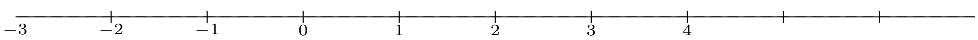
   Solution: One function that works if $r(x) = \frac{2(x^2-9)}{x(x-1)}$.

   (d) Build the sign chart for your function. Finally, use all the information to sketch the graph of f.

   Solution: The places where the sign of r can change are $x = \pm 3, x + 0$ and $x = -1$. In fact, r does change signs at all these places.



(e) From the graph, you can speculate on the existence of critical points if there are any. Write a sentence about where you expect to find these critical points or why you think there are none. Estimate the sign chart for $r'(x)$



Solution: Based on the graph, r has one critical point and it is in the interval $(0, 1)$. Suppose it is α. The f' is negative on $(-\infty, -3)$, negative on $(\alpha, 3)$, and negative on $(3, \infty)$.

5. (20 points) Find a fourth degree polynomial function $p(x)$ which has the following properties:

- p has relative minima at $x = -2$ and $x = 2$ and a relative maxima at $x = 0$,
- $p(0) = 4$. Hint: how must the function $p'(x)$ factor in order to have the three critical points $x = 0, x = \pm 2$?

Solution: First note that $p'(x) = x(x^2-4) = x^3-4x$. So $p(x) = x^4/4+2x^2+c$ for some constant c. Since $p(0) = 4$, we have $c = 4$. So $p(x) = x^4/4 - 2x^2 + 4$.

6. (10 points) The line tangent to the graph of $g(x)$ at the point $(4, 6)$ has a y-intercept of 9. What is $g'(4)$?

   Solution: The line has slope $(9 - 6)/(0 - 4) = -3/4$.

7. (10 points) Find all the points (x, y) on the graph of $h(x) = 2x^2 - 4x$ where the tangent line has a slope equal to 5.

   Solution: Since $h'(x) = 4x - 4$, we can solve $4x - 4 = 5$ for x, which yields $x = 9/4$ and $y = 2 \cdot 81/16 - 36/4 = 18/16 = 9/8$.

8. (48 points) Find the following antiderivatives.

   (a) $\int 2x - 5\ dx$

   Solution: $x^2 - 5x + C$.

   (b) $\int 9x^2 - 4x - 2/x\ dx$

   Solution: $3 \cdot x^3 - 2 \cdot x^2 - 2 \ln x + C$.

   (c) $\int \frac{\ln(x)}{x}\ dx$

   Solution: Let $u = \ln(x)$. Then get $(\ln(x))^2/2 + C$.

   (d) $\int x(x-1)(x-2)^5\ dx$

   Solution: Let $u = x - 2)$.

   (e) $\int \frac{3x^3 + 2x^2 - x}{x}\ dx$

   Solution: $\int (3x^3 + 2x^2 - x)/x\ dx = \int 3x^2 + 2x - 1 dx = x^3 + x^2 - x + C$.

   (f) $\int \frac{2x+3}{x^2+3x-3}\ dx$

   Solution: By substitution, $(u = x^2 + 3x - 3)$, $\int \frac{2x+3}{x^2+3x-3}\ dx = \ln|x^2 + 3x - 3| + C$.

   (g) $\int 6x^5(x^6+3)^7\ dx$

   Solution: By substitution with $u = x^6+3$, $\int 6x^4(x^6+3)^7\ dx = \frac{(x^6+3)^8}{8} + C$.

   (h) $\int x^2 e^{x^3}\ dx$

   Solution: By substitution with $u = x^3, du = 3x^2$, $\int e^u\ du = e^u + C = (1/3)e^{x^3} + C$.

   (i) $\int \frac{2x+3}{x-1}\ dx$. Try substitution with $u = x - 1$.

   Solution: By substitution with $u = x - 1, du = dx$, $\int \frac{2u+5}{u}\ du = \int 2 + \frac{5}{u}\ du = 2u + 5 \ln u = 2((x-1) + 5\ln(x-1) + C$.

   (j) $\int (x+2)^4(x-2)\ dx$

   Solution: By substitution with $u = x+2$, $\int (x+2)^4(x-2)\ dx = \int u^4(u - 4)\ du = \int u^5 - 4u^4\ du = u^6/6 - 4u^5/5 + C = (x+2)^6/6 - 4(x+2)^5/5 + C$.

9. (32 points) Find the following integrals.

   (a) $\int_0^1 \sqrt{5x+4}\ dx$
   Solution: Let $u = 5x + 4$.

   (b) $\int_0^1 \frac{e^x}{e^x+1}\ dx$
   Solution: Let $u = e^x + 1$.

   (c) $\int_1^6 |x-3|\ dx$
   Solution: Use geometry.

   (d) $\int_0^1 e^{\ln(x^3)}\ dx$
   Solution: Note that $e^{\ln(x^3)} = x^3$.

10. (10 points) Suppose you're given a function f to differentiate. You do so and you get $f'(x) = (x+3)(x+1)(x^2)(x-4)$. You conclude that f has four critical points, $x = -3, -1, 0$ and 4. Classify each of these as a) the location of a relative maximum, b) the location of a relative minimum, or c) an imposter.

    Solution: Because there is no sign change at $x = 0$, we know 0 is an imposter. Since $f'(x)$ changes from negative to positive at both -3 and 4, we know thatthere are minimums is the location of a minimum. There is a relative max at $x = -1$

11. (20 points) Consider the function $g(x) = \ln((x-3)(x^2+5x+4))$. Notice that $g(0) = \ln(-3 \cdot 4)$ is not defined because x must be positive in order to take ln of it. Find the domain of $g(x)$ and put your answer in interval notation. Then find the derivative of g and the critical points.

    Solution: Factor $p(x) = (x-3)(x^2+5x+4) = (x-3)(x+1)(x+4)$, so the domain of g is the union of the two intervals where p is positive, $(-4,-1) \cup (3, \infty)$. The derivative of g is $\frac{1}{x-3} + \frac{1}{x+1} + \frac{1}{x+4}$. g has a critical point in the interval $(-4, -1)$.

12. (20 points) Consider the five functions $f(x) = 3x + 4$, $g(x) = (3x - 2)^2$, $h(x) = \ln(x^5 + 1)$, $j(x) = e^{2x}$, and $k(x) = x^3 - x^2$.

   (a) Which of the functions is growing fastest at $x = 2$.

   (b) Put the functions in order from least fastest growth at $x = 2$ to the fastest growth at $x = 2$.

   Solution: Both here and the part below, the order is $h < f < k < g < j$.

   (c) Put the functions in order from least to greatest by their growth over the interval $[1, 6]$.

13. (12 points) Let $H(x) = 2x - \ln(4x^2 + 12x + 10)$. Find all the critical points.

   Solution: We need to solve the equation $\frac{8x+12}{4x^2+12x+10} = 2$. This is equivalent to $8x^2 + 16x + 8 = 0$ which has repeated roots, $x = -1$.

14. (20 points) Consider the function $f(x) = (2x - 4)e^{x^2}$.

   (a) Use the product rule to find $f'(x)$.

   Solution: $f'(x) = 2e^{x^2} + 2x(2x - 4)e^{x^2}$

   (b) List the critical points of f.

   Solution: Factor the expression above to get $(4x^2-8x+2)e^{x^2} = 2e^{x^2}(2x^2-4x+1)$, which has value 0 when $x = 1 - \sqrt{2}/2 \approx 0.293$, $x = 1 + \sqrt{2}/2 \approx 1.707$. Call the first value α and the second β.

   (c) Construct the sign chart for $f'(x)$.

   Solution: f' is positive on $(-\infty, \alpha)$ and on (β, ∞).

   (d) Write in interval notation the interval(s) over which f is increasing.

   Solution: f is increasing on $(-\infty, \alpha)$ and on (β, ∞).

15. (15 points) Find a number b such that $\displaystyle\int_b^{2b} x^3\,dx = 60$.

   You must show your work to get any credit on this problem. Guessing b is not enough.

   Solution: $x^4/4|_b^{2b} = \frac{1}{4}(16b^4 - b^4) = \frac{15}{4}b^4 = 60$ and it follows that $b = 2$.

16. (15 points) The region R is bounded by the vertical lines $x = 1$ and $x = e^2$ and by the graph of $f(x) = 1 + \frac{1}{x}$ and the x-axis. Find the area of R.

   Solution: Antidifferentiate to get $x + \ln(x)|_1^{e^2} = 2 + e^2 - 1 = 1 + e^2$.

17. (30 points) Let $A = (6, 6)$, $B = (1, 4)$, $C = (1, 0)$ and $D = (6, 0)$ be the vertices of a quadrilateral in the plane.

    (a) Sketch the figure and use geometry to find the area of $ABCD$

    Solution: The area is 25.

    (b) Find an equation for the linear function (the line) that goes through the points A and B. Give this function the name f.

    Solution: Since we know two points on the line, it follows that $f(x)-4 = \frac{6-4}{6-1}(x-1)$, which we can write as $f(x) = 2x/5 + 18/5$.

    (c) Use calculus to find the area of the region R defined as follows:

    $$R = \{(x, y) : 1 \leq x \leq 6,\ 0 \leq y \leq f(x)\}$$

    Solution: We need to find $\int_1^6 2x/5+18/5\, dx$. This is just $\left.\frac{x^2+18x}{5}\right|_1^6 = 25$.

18. (10 points) Recall that a zero over zero problem is a limit problem of the form $\lim_{x\to a} \frac{f(x)}{g(x}$, where both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are zero. Build a zero over zero problem

    (a) whose limit is 3.

    Solution: One solution is $f(x) = 3x$ and $g(x) = x$

    (b) whose limit is 0.

    Solution: One solution is $f(x) = x^2$ and $g(x) = x$

    (c) whose limit fails to exist.

    Solution: One solution is $f(x) = 3x$ and $g(x) = x^2$