December 13, 2017 **Name** ____________________

The total number of points available is 274. Throughout this test, **SHOW YOUR WORK.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (30 points) Domain Problems.
   For each function listed below, find the (implied) domain. Write your answer in interval notation.

   (a) $f(x) = \frac{x}{x^2+5x+6}$.
   Solution: Rewrite the denominator as $(x+2)(x+3)$. We want the denominator not to be zero, so we must 'pluck' out the two numbers $x = -2$ and $x = -3$. Thus, the domain is $(-\infty, -3) \cup (-3, -2) \cup (-2, -\infty)$

   (b) $g(x) = \sqrt{16 - \sqrt{x}}$.
   Solution: We need to solve the inequality $16 - \sqrt{x} \geq 0$. So $\sqrt{x} \leq 16$. Square both sides to get $x \leq 256$. Of course we must also have $0 \leq x$ so the $\sqrt{x}$ is defined. Thus we have $[0, 256]$ as the domain.

   (c) $h(x) = \ln((x-3)(x-1)(x+3))$ Notice that $h(0) = \ln((-3)(-1)(3) = \ln 9$, so 0 belongs to the domain of h.
   Solution: Solve the inequality $(x-3)(x-1)(x+3) > 0$ to get $(-3, 1) \cup (3, \infty)$.

   (d) $k(x) = \sqrt{|x-1| - 3}$.
   Solution: Solve the equality $|x-1| - 3 = 0$ to get $x = -2$ and $x = 4$, so we have $(-\infty, -2] \cup [4, \infty)$.

   (e) $G(x) = \sqrt{\ln(x) - 1}$.
   Solution: The domain is $[e, \infty)$.

2. (30 points) Limit Problem

   (a) Find $\lim\limits_{x \to -1} \dfrac{x^3 - x^2 + x + 3}{x^2 - 1}$.
   Solution: To resolve the zero over zero conflict, divide $x^3 - x^2 + x + 3$ by $x + 1$ to get $x^2 - 2x + 3$, then take the limit of the quotient obtained by eliminating the common factor $x + 1$. $\lim_{x \to 1} \frac{x^2-2x+3}{x-1} = 6/-2 = -3$.

   (b) Suppose $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

      i. Is it possible that $\lim_{x \to a} f(x) \cdot g(x) = 3$?
      Solution: The limit of the product is the product of the limits, so $\lim_{x \to a} f(x) \cdot g(x) = 0$.

ii. Is it possible that $\lim_{x\to a} f(x)/g(x) = 4$?
Solution: Yes, suppose $f(x) = 4x$ and $g(x) = x$. Then $\lim_{x\to a} f(x)/g(x) = 4$.

iii. What are the possible outcomes of $\lim_{x\to a} \frac{f(x)}{g(x)}$? Can this limit fail to exist? Must the limit fail to exist? Write a sentence or two to show that you understand this question.
Solution: This limit can fail to exist. It can also be any real number.

3. (20 points) A function $g(x)$ has been differentiated to get

$$g'(x) = 2(x-5)^2 - 8.$$

(a) Find the interval(s) over which $g(x)$ is increasing.
Solution: Solve $2(x-5)^2 - 8 = 0$ to find the two critical points of g, $x = 3$ and $x = 7$, and then build the sign chart for g' to see that it's negative precisely on $(3, 7)$, so g is increasing on $(-\infty, 3)$ and $(7, \infty)$.

(b) Find the interval(s) over which $g'(x)$ is increasing.
Solution: Since $g'(x)$ is a concave up quadratic polynomial with vertex $(5, -8)$, we conclude that g' is increasing on $(5, \infty)$.

(c) Find the interval(s) over which $g(x)$ is concave upwards.
Solution: Differentiate g' to get $g''(x) = 2 \cdot 2(x-5)$ which is positive on $(5, \infty)$, so that is the interval where g is concave upwards.

(d) Find a representation of the function g.
Solution: Since $g'(x) = 2(x^2 - 10x + 25) - 8 = 2x^2 - 20x + 42$, it follows that one g could be $g(x) = 2x^3/3 - 10x^2 + 42x$.

4. (30 points) There is a cubic polynomial $p(x)$ with zeros at $x = -2, x = 1$, and $x = 2$.

(a) Build one such function.
Solution: $f(x) = (x+2)(x-1)(x-2)$ is such a function.

(b) Build the sign chart for your function.
Solution: The function above is positive on $(-2, 1) \cup (2, \infty)$.

(c) Find an interval over which your function is increasing?
Solution: Build the sign chart for $f'(x) = 3x^2 - 2x - 4$. I'm getting $\frac{2\pm2\sqrt{13}}{6}$.

(d) Find the area of the region bounded by your function over the interval from $x = -2$ to $x = 1$.

Solution: First $f(x) = (x^2 - 4)(x - 1) = x^3 - x^2 - 4x + 4$, which is positive over $(-2, 1)$, so the area caught underneath the graph of f is $\int_{-2}^{1} x^3 - x^2 - 4x + 4dx = x^4/4 - x^3/3 - 2x^2 + 4x|_{-2}^{1} = 1/4 - 1/3 - 2 + 4 - (4 + 8/3 - 8 - 8) = 45/4$.

5. (12 points) Given $f''(x) = 2x - 6$ and $f'(-2) = 6$ and $f(-2) = 1$. Find $f'(x)$ and $f(x)$.

Solution: First write $f'(x) = x^2 - 6x + C$ by the power rule. Solve $f'(-2) = 6$ for C to get $C = -10$. Then $f'(x) = x^2 - 6x - 10$. Therefore $f(x) = \int x^2 - 6x - 10\ dx = x^3/3 - 3x^2 - 10x + C$. We can solve $f(-2) = 1$ to get $C = -13/3$, so the function is $f(x) = x^3/3 - 3x^2 - 10x - 13/3$.

6. (12 points) Let $f(x) = \dfrac{3}{x} - 2e^x$.

(a) Find an antiderivative of $f(x)$.

Solution: Note that $\int \frac{3}{x} - 2e^x\ dx = 3 \ln x - 2e^x$.

(b) Compute $\displaystyle\int_1^e f(x)dx$.

Solution: Note that $\int \frac{3}{x} - 2e^x\ dx = 3 \ln x - 2e^x$. So $\displaystyle\int_1^e f(x)dx = 3 \ln x - 2e^x|_1^e = 3 \ln e - 2e^e - (3 \ln 1 - 2e) = 3 - 2e^e + 2e \approx -18.872$.

7. (42 points) Compute each of the following integrals

(a) $\displaystyle\int_1^2 \frac{(4x - 5)^2}{x}\ dx$

Solution: Rewrite the integrand to get $\displaystyle\int_1^2 \frac{(4x - 5)^2}{x}\ dx = \int_1^2 \frac{16x^2 - 40x + 25}{x}\ dx = \int_1^2 16x - 40 + \frac{25}{x}\ dx$. Thus, we have $8x^2 - 40x + 25 \ln(x)|_1^2 \approx 1.328$.

(b) $\displaystyle\int_0^1 \frac{d}{dx}(x^3 - 2x^2 + 7)\ dx$

Solution: This is just $x^3 - 2x^2 + 7|_0^1 = -1$.

(c) $\displaystyle\int_1^4 3x^2 e^{x^3}\ dx$

Solution: $e^{x^3}|_1^4 = e^{64} - e^1$.

(d) $\displaystyle\int_2^3 \frac{x^3+2x^2-x}{x}\,dx$

Solution: $\int (x^3+2x^2-x)/x\;dx = \int x^2+2x-1dx = x^3/3+x^2-x|_2^3 = 31/3$.

(e) $\displaystyle\int_1^3 \frac{2x+3}{x^2+3x-3}\,dx$

Solution: By substitution, $(u = x^2+3x-3)$, $\int \frac{2x+3}{x^2+3x-3}\;dx = \ln|x^2+3x-3||_1^3 = \ln(15) \approx 2.71$.

(f) $\displaystyle\int_{-1}^1 6x^5(x^6+3)^7\,dx$

Solution: By substitution with $u = x^6+3$, $\int 6x^5(x^6+3)^7\;dx = \frac{(x^6+3)^8}{8}|_{-1}^1 = 0$.

(g) $\displaystyle\int_1^2 (x-1)^9 x\,dx$

Solution: By substitution with $u = x-1, du = dx$, $\int (x-1)^9 x\;dx = \int u^9(u+1)\;du = \int u^{10}+u^9\;du = \frac{1}{11}u^{11}+\frac{1}{10}u^{10} = \frac{1}{11}(x-1)^{11}+\frac{1}{10}(x-1)^{10}|_1^2 = \frac{1}{11}+\frac{1}{10} = \frac{21}{110}$.

8. (15 points) Find the intervals over which $f(x) = x^2e^{2x}$ is increasing.

   Solution: First $f'(x) = 2xe^{2x} + x^2\cdot 2e^{2x} = 2e^{2x}(x+x^2)$, so we need to solve $x+x^2 = 0$ and we get $x = 0$ and $x = -1$. Since f' is negative precisely on $(-1, 0)$, f is increasing on $(-\infty, -1)$ and on $(0, \infty)$.

9. (12 points) Is there a value of b for which $\displaystyle\int_b^{2b} x^4+x^2\,dx = 128/15$? If so, find it.

   Solution: Use the power rule to get the equation $(2b)^5/5+(2b)^3/3-(b^5/5+b/3) = 128/15$. It follows that $b = 1$.

10. (20 points) Use the substitution technique to find $\int (x-2)^4\cdot x\;dx$. Then differentiate to check your answer.

    Solution: Let $u = x-2$, then $du = dx$ and $\int (x-2)^4\cdot x\;dx = \int u^4(u+2)\;du$, and this give rise to $\frac{u^6}{6}+2\frac{u^5}{5}+C = \frac{(x-2)^6}{6}+\frac{2(x-2)^5}{5}+C$.

11. (10 points) Find an interval where the function g defined by $g(x) = \ln(e^{x^2-4x})$ is increasing.

    Solution: Since $g(x) = \ln(e^{x^2-4x}) = x^2-4x$, it follows that $g'(x) = 2x-4$ and so G is increasing on $(2, \infty)$.

12. (10 points) Compute $\int \frac{d}{dx}\, xe^{x^2}\, dx$.

    Solution: One function whose derivative is $\frac{d}{dx}\, xe^{x^2}$ is xe^{x^2}.

13. (15 points) Suppose x and y are positive real numbers satisfying $2xy = 9$.

    (a) Find two pairs of numbers (x_1, y_1) and (x_2, y_2) satisfying the condition $2xy = 9$. Compute the value of $2x + 3y$ for each of these pairs.

    Solution: Two possible points are $(1, 9/2)$ and $(2, 9/4)$, where the values are 15.5 and 10.75 respectively.

    (b) What is the smallest possible value of $2x + 3y$ such that $2xy = 9$.

    Solution: Minimize the function $f(x) = 2x + 27/2x$ to get $x = 3\sqrt{3}/2$.

    (c) What is the smallest possible value of $3x + 4y$ such that $2xy = 9$.

    Solution: This is similar to the one above.

14. (20 points) Use calculus to find the area of the trapezoid R bounded above by the graph of $f(x) = 2x + 1$, below by the x-axis, and on the sides by $x = 1$ and $x = 5$.

    Solution: $\int_1^5 2x + 1\; dx = x^2 + x|_1^5 = 5^2 + 5 - (1^2 + 1) = 30 - 2 = 28$.

15. (20 points) Use calculus to find the area of the trapezoid R bounded above by the graph of $f(x) = 6 - x/3$, below by the x-axis, and on the sides by $x = 0$ and $x = 9$. Then find the area without using calculus to check your answer. SHOW ALL YOUR WORK

    Solution: $\int_1^5 2x + 1\; dx = x^2 + x|_1^5 = 5^2 + 5 - (1^2 + 1) = 30 - 2 = 28$.