Math 1120 **Calculus** **Final Exam**

May 6, 2002 Your name ____________________

As usual, show all your work. If you used a calculator, explain in detail how you used it to solve the problem. Part A is worth 120 points and part B 130 points.

1. (40 points) Find the following antiderivatives.

   (a) $\int 6x^3 - 5x - 1dx$

   Solution: $3/2 \cdot x^4 - 5/2 \cdot x^2 - x + c$.

   (b) $\int 6x^{\frac{3}{2}} + x^{-\frac{1}{2}}dx$

   Solution: $6 \cdot 2/5 \cdot x^{5/2} + 2x^{1/2} + c$.

   (c) $\int \frac{3x^3 + 2x - 1}{x}dx$

   Solution: $\int 3x + 2 - 1/xdx = 3x^2/2 + 2x - \ln|x| + c$.

   (d) $\int \frac{2x+1}{x^2+x-3}dx$

   Solution: By substitution, $(u = x^2+x-3)$, $\int \frac{2x+1}{x^2+x-3}dx = \ln|x^2+x-3|+c$.

   (e) $\int 5x^4(x^5+2)^7dx$

   Solution: By substitution with $u = x^5 + 2$, $\int 5x^4(x^5+2)^7dx = \frac{(x^5+2)^8}{8} + C$.

2. (20 points) Compute the following definite integrals.

   (a) $\int_0^2 2xe^{-x^2}dx$

   Solution: $-e^{-x^2}]_0^2 = 1 - e^{-4} \approx 0.9816$.

   (b) $\int_0^5 (2x-1)\sqrt{x^2 - x + 5}\,dx$

   Solution: $2/3(x^2 - x + 5)^{3/2}]_0^5 = \frac{10}{3}(25 - \sqrt{5}) \approx 75.8798$.

3. (15 points) Find a function $G(x)$ whose derivative is $3x^2 - 7$ and for which $G(4) = 9$.

   Solution: $G(x) = x^3 - 7x + C$ for some constant C. But since $G(4) = 9 = 4^3 - 28 + C$, it follows that $C = -36 + 9 = -27$ and $G(x) = x^3 - 7x - 27$.

4. (15 points) Find the area of the region bounded by $y = x^{3/2}$, the x-axis, and the lines $x = 0$ and $x = 4$.

Solution: The area is given by $\int_0^4 x^{3/2}dx = \frac{2}{5}x^{5/2}|_0^4 = \frac{2}{5}(4^{5/2} - 0^{5/2}) = \frac{2}{5}\cdot 32 = \frac{64}{5}$.

5. (30 points) An object is thrown upwards from the top of a 400 feet high building, after which its path is governed entirely by gravity. The acceleration due to gravity is given by $a(t) = -32$, where t is measured in seconds and $a(t)$ is measured in ft/sec^2. Recall that $a(t) = v'(t) = s''(t)$, where $v(t)$ denotes the velocity of the object (negative when its moving towards the earth), and $s(t)$ is the position of the object at time t. The equation above simply says that the velocity is an antiderivative of acceleration, and that position is an antiderivative of velocity. Suppose the initial velocity is 80 feet per second; ie, $v(0) = 80$. Answer the following questions about the path of the object.

   (a) Compute the function $v(t)$.
   Solution: $v(t) = -32t + C = -32 + 80$ since $V(0) = 80$.

   (b) Compute the function $s(t)$.
   Solution: $s(t) = -16t^2 + 80t + C = -16t^2 + 80t + 400$ because $s(0) = 400$.

   (c) At what time does the object hit the ground?
   Solution: Solve the equation $16t^2 - 80t - 400 = 0$ to find that $t = \frac{5 \pm \sqrt{25+100}}{2}$, only one of which is positive. Thus, $t = \frac{5+5\sqrt{5}}{2} \approx 8.09$.

   (d) At what time does the object reach its maximum height?
   Solution: $v'(t) = -32t + 80 = 0$ happens at $t = 2.5$

   (e) What is its maximum height?
   Solution: $s(2.5) = -16(2.5)^2 + 80(2.5) + 400 = 574.70$ ft.

Part B

1. (5 points) The slope of the line that contains the points $(-1, y)$ and $(4, -12)$ is -3? What is y?

   Solution: Solve the equation $\frac{y+12}{-1-4} = -3$ to get $y = 3$.

2. (5 points) What is the slope of the line perpendicular to the line $2y + x = 4$?

   Solution: The slope of the given line is $-1/2$ so the slope of its perpendicular is 2.

3. (10 points) Over what intervals is the second derivative of $g(x) = x^4 - 6x^3 + 12x^2 + 2x + 2$ negative?

   Solution: Since $g'(x) = 4x^3 - 18x^2 + 24x + 2$, it follows that $g''(x) = 12x^2 - 36x + 24 = 12(x-1)(x-2)$. Use the Test Interval Technique to determine that g'' is negative on the interval $(1, 2)$.

4. (15 points) Construct a cubic polynomial $f(x)$ that has zeros at $x = -2$, $x = 1$, and $x = 3$ and satisfies $f(0) = -12$.

   Solution: Since the zeros are $-2, 1$, and 3, the function must be $f(x) = a(x+2)(x-1)(x-3)$ for some constant a. Then $f(0) = a(2)(-1)(-3) = 6a = -12$, so $a = -2$, and $f(x) = -2(x+2)(x-1)(x-3)$.

5. (10 points) Let $g(x)$ be defined as follows: Let

$$g(x) = \begin{cases} e^x & \text{if } x \leq 1 \\ \ln(x) & \text{if } x > 1 \end{cases}$$

   Find an equation for the line tangent to the graph of $g(x)$ at the point $(3, g(3))$.

   Solution: First note that $g'(3) = 1/3$ and that $g(3) = \ln(3)$ so the line is $y - \ln(3) = \frac{1}{3}(x-3)$, which simplifies to $y = x/3 - 1 + \ln(3)$.

6. (20 points) Compute each of the following derivatives.

   (a) $\frac{d}{dx}\frac{\sqrt{x^2-3}}{2x}$

   Solution: By the quotient rule, $\frac{d}{dx}\frac{\sqrt{x^2-3}}{2x} = \frac{x^2(x^2-3)^{-1/2} - 2(x^2-3)^{1/2}}{4x^2}$, which simplifies slightly.

   (b) $\frac{d}{dx}e^{x+\ln(x)}$

   Solution: By the chain rule, $\frac{d}{dx}e^{x+\ln(x)} = e^{x+\ln(x)}(1 + \frac{1}{x})$.

   (c) $\frac{d}{dx}\ln(x^2 + e^{2x})$

   Solution: By the chain rule, $\frac{d}{dx}\ln(x^2 + e^{2x}) = \frac{2x+2e^{2x}}{x^2+e^{2x}}$.

7. (10 points) Calculate the doubling time for a 7% investment compounded continuously.

   Solution: Solve $2P = Pe^{rt}$ where $r = 0.07$ to get $2 = e^{0.07t}$ or $t = \frac{\ln(2)}{0.07} \approx 9.902$ years.

8. (30 points) Suppose $u(x)$ is a function whose derivative $u'(x) = (2x+1)^2(x-2)^2$. Recall that theorem B tells you the intervals over which $u(x)$ is concave upwards based on $u''(x)$.

   (a) Compute $u''(x)$.
   Solution: $u''(x) = 2(2x+1)2(x-2)^2 + (2x+1)^2 \cdot 2(x-2)$
   $= 2(2x+1(x-2)[2(x-2)+(2x+1)] = 2(2x+1)(x-2)(4x-3)$.

   (b) Find the three zeros of $u''(x)$.
   Solution: $x = -1/2, x = 2$, and $x = 3/4$.

   (c) Use the Test Interval Technique to find the intervals over which $u(x)$ is concave up.
   Solution: u is concave up over $(-1/2, 3/4)$ and over $(2, \infty)$.

9. (5 points) What is $\lim_{h\to 0} \dfrac{1-\sqrt{1+2h}}{h}$?

   Solution: Rationalize the numerator to get $\lim h \to 0 \frac{-2}{1+\sqrt{1+2h}} = -1$.

10. (20 points) Find the absolute maximum and absolute minimum of the function $h(x) = \sqrt{x^2+6x+25}$ over the interval $[-5, 5]$.

    Solution: Differentiate to get $h'(x) = (2x+6)(x^2+6x+25)^{-1/2}$ so the only critical point is $x = -3$. Checking endpoints, we have $h(-5) = \sqrt{20}$, $h(-3) = \sqrt{16}$ and $h(5) = 4\sqrt{5}$. So the minimum value is 4 which occurs at $x = -3$ and the maximum value is $4\sqrt{5}$ which occurs at $x = 5$.