May 7, 2014 **Name** ______________________________

Each of the first 17 problems are worth 10 points. The other problems are marked. The total number of points available is 285. Throughout the free response part of this test, to get credit you must **show your work.**

1. Let $f(x) = x^3 - 2x + 4$. Find an equation for the line tangent to the graph of f at the point $(1, f(1))$.

   Solution: $f'(x) = 3x^2 - 2$, so $f'(1) = 1$. Therefore the line is $y - 3 = 1(x - 1)$.

2. Consider the function $f(x) = xe^{2x}$. Find a critical point. Does f have a relative maximum, a relative minimum, or neither at that point?

   Solution: By the product rule, $f'(x) = xe^{2x} \cdot 2 + e^{2x} = e^{2x} = e^{2x}(2x + 1)$, so $x = -1/2$ is the only critical point.

3. What is $\lim_{x\to\infty} \dfrac{(x-2)(2x-3)}{(3x+2)(4x-1)}$?

   Solution: Using the asymptote theorem, $\lim_{x\to\infty} \dfrac{(x-2)(2x-3)}{(3x+2)(4x-1)} =$ $2/12 = 1/6$.

4. What is $\lim_{x\to 2}\dfrac{x^3-8}{x^2-4}$?

   Solution: Factor both numerator and denominator to get $\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = 12/4 = 3$

5. What is $\lim_{x\to 4}\dfrac{\sqrt{x}-2}{x^2-16}$?

   Solution: Rationalize the numerator and also factor the denominator to see that $x-4$ is a factor of both numerator and denominator. Then take limit to get $1/32$.

6. Let $F(x)$ be an antiderivative of $3x^2-2x$. What is the growth of $F(x)$ over the interval $[1,5]$?

   Solution: One antiderivative is $F(x) = x^3 - x^2$ which grows from 0 to 100 on the given interval.

7. Let $H(x) = \ln(4x^2+12x+10) - 2x$. Find a critical point.

   Solution: We need to solve the equation $\frac{8x+12}{4x^2+12x+10} = 2$. This is equivalent to $8x^2+16x+8 = 0$ which has repeated roots, $x = -1$

8. Let $g(x) = 2x^3 - 7x^2 + 4x - 10$. Find the intervals over which g is increasing?

   Solution: $g'(x) = 6x^2 - 14x + 4 = 2(3x^2 - 7x + 2) = 2(3x - 1)(x - 2) > 0$ on $(-\infty, 1/3))$ and $(2, \infty)$.

9. Let $k(x) = 2x^4 - 14x^3 + 30x^2 + 10x$. Find the intervals over which k is concave upwards?

   Solution: $k''(x) = 12(2x - 5)(x - 1) < 0$ on $(1, 5/2)$, so k is concave upwards on $(-\infty, 1)$ and on $(5/2, \infty)$.

10. What is the value of $\int_2^5 \frac{d(2x-5)^4}{dx}\ dx$?

    Solution: Since differentiation and antidifferentiation just undo each other, its just $(2x - 5)^4|_2^5 = 5^4 - (-1)^4 = 625 - 1 = 624$.

11. Find all of the zeros of the polynomial $p(x) = (x-1)^3(x+2)^2 - 4(x-1)^2(x+2)$.

    Solution: Factor p to get $p(x) = (x - 1)^2(x + 2)[(x - 1)(x + 2) - 4]$. Factor the result $x^2 - x - 6$ to get the two zeros $x = -2$ and $x = 3$.

12. How many solutions does the equation $|x^2 - 4| = 3$ have? Find them all.

    Solution: There are 4 solutions, $x = \pm 1$, $x = \pm\sqrt{7}$.

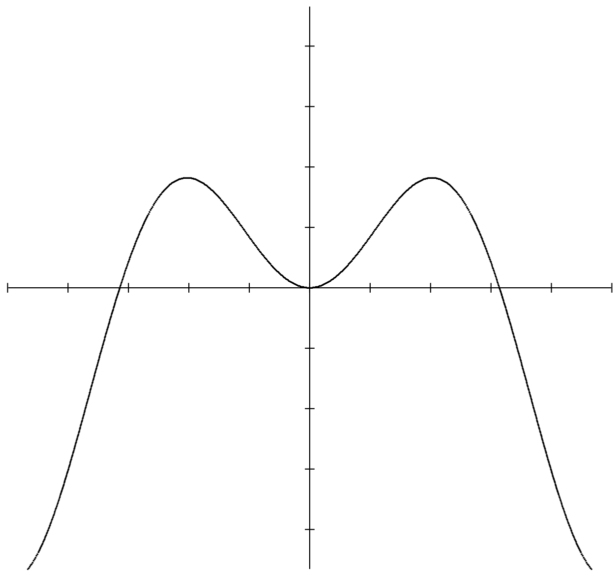
13. Let $h(x) = x^3(x^2 - 1)^2$. Find all the critical points of h.

    Solution: $h'(x) = 3x^2(x^2 - 1)^2 + 2(x^2 - 1) \cdot 2x \cdot x^3 = x^2(x^2 - 1)[3(x^2 - 1) + 4x^2]$ which has zeros $x = \pm 1, x = 0$, and $x = \pm\sqrt{3/7}$.

14. Find a value of t that satisfies $\int_{-t}^{t} x^3 + x^2 \, dx = 18$.

    Solution: The integral has value $x^4/4 + x^3/3|_{-t}^{t} = 2t^3/3 = 18$, so $t^3 = 27$. It follows that $t = 3$.

15. The next three problems may have multiple correct answers. Circle **all** the correct answers. The three problems all refer to the graph of the function $g'(x)$ given below.



(a) The function g has a relative maximum or minimum

(A) near -3 **(B)** at $x = -2$ **(C)** at $x = 0$

(D) at $x = 2$ **(E)** near 3

Solution: A. and E. The critical point 0 is an imposter. The other two are real.

(b) The function g has an inflection point (ie, a change in concavity)

(A) near -3 **(B)** at $x = -2$ **(C)** at $x = 0$

(D) at $x = 2$ **(E)** near 3

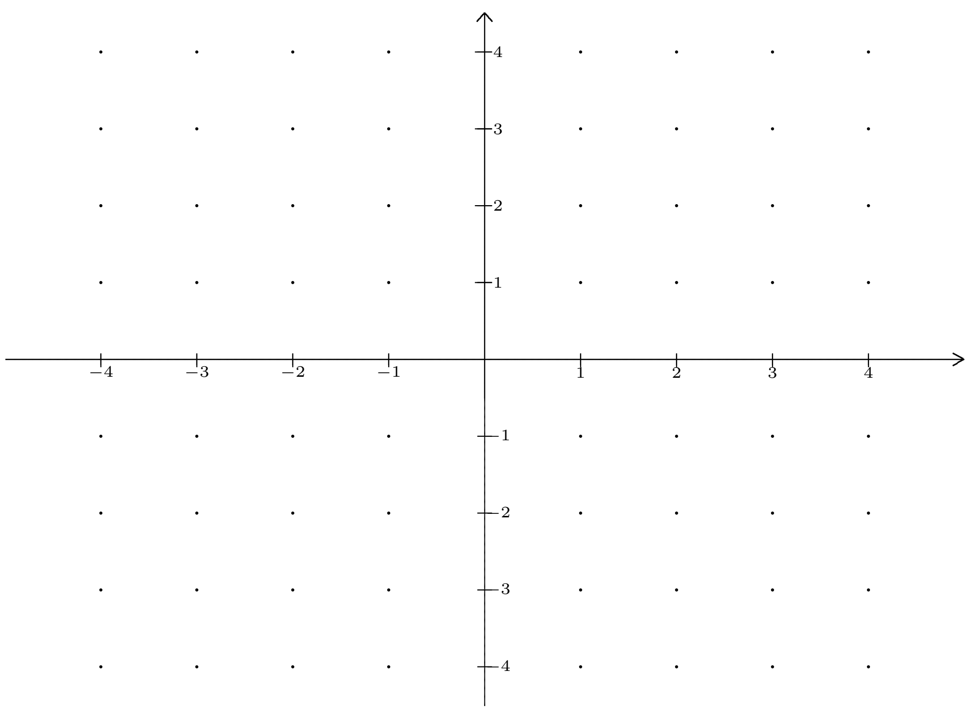
Solution: The function $g''(x)$ changes signs at $x = -2, 0$ and 2.

(c) The function g increases over

(A) $[-5, -2]$ **(B)** $[-3, 0]$ **(C)** $[0, 3]$ **(D)** $[-2, 2]$ **(E)** $[2, 3]$

Solution: B, C, D, E. g increases over each interval where $g'(x) \geq 0$, so roughly any interval with both endpoints in $[-3, 3]$.

16. (25 points) Consider the rational function $r(x) = \frac{(2x-4)(x^2-9)(x+3)}{(x^2-4)(x+1)(x)}$. Find the vertical and horizontal asymptotes, and the zeros of the function. Build the sign chart for the function. Sketch the graph on the coordinate system provided. Use the graph to state the intervals over which the function is increasing. Using the graph you find, build the sigh chart for the derivative function $r'(x)$.



Solution: The function reduces to $r(x) = \frac{2(x-3)(x+3)^2}{(x+2)(x+1)(x)}$. From here we can read off the zeros: $x = -3$ and $x = 3$; the vertical asymptotes: $x = -2$ and $x = -1$ and $x = 0$, and the horizontal asymptote $y = 2$. The function is increasing over each of the intervals $(-\infty, -3)$ and $(\alpha, \beta))$ where $\alpha \approx -3/2$ and $\beta \approx -1/2$. From this we can build the sign chart for $r'(x)$.

17. (20 points) Find the area caught between the graphs of the two functions $f(x) = -(x-1)(x-3)$ and $g(x) = x - 3$.

    Solution: The two functions agree at $x = 0$ and $x = 3$, so the area is $\int_0^3 f(x) - g(x)\ dx == \int_0^3 3x - x^2\ dx = 3x^2/2 - x^3/3|_0^3 = 27/2 - 9 = 9/2$.

18. (20 points) Among all the lines tangent to the graph of $f(x) = 6x^2 - x^3$ over the interval $(0, 6)$, which one has the greatest slope. Find an equation for the one with the greatest slope.

    Solution: We are trying to maximize the function $f'(x) = 12x - 3x^2 = 3x(4-x)$, which is a parabola that opens downward. The zeros of the parabola are $x = 0$ and $x = 4$, so the vertex is $(2, f(2))$. Alternatively, note that $f''(x) = 12 - 6x$, which is zero at $x = 2$. Since $f'(2) = 3 \cdot 2(4-2) = 12$ and $f(2) = 6 \cdot 2^2 - 2^3 = 16$, an equation for the tangent line with largest slope is $y - 16 = 12(x - 2)$.

19. (30 points) Consider the curve $y = x^2$, your favorite parabola. In this problem you are asked to find the point on the curve that is closest to the point $(18, 0)$. In case you missed this problem when it appeared on set 8, the steps below will enable you to solve it.

    (a) Note that $(0, 0)$ belongs to the curve and the distance from $(18, 0)$ to $(0, 0)$ is 18. Find two other points on the curve and compute their distance from $(18, 0)$.

    Solution: $D((1, 1), (18, 0)) = \sqrt{17^2 + 1^1} = \sqrt{290}$, and $D((2, 4), (18, 0)) = \sqrt{16^2 + 4^2} = \sqrt{260}$. Of course any other two points on the curve could have been chosen as well.

    (b) Note that all the points on the curve are of the form (x, x^2) for some real number x. Suppose the point on the curve closest to $(18, 0)$ is (a, a^2). What is the slope, in terms of a, of the line tangent to the curve at the point (a, a^2)?

    Solution: Since the derivative is $2x$, the slope of the tangent line is $2a$.

    (c) What is the slope, in terms of a, of the line connecting (a, a^2) with $(18, 0)$. There are two ways to get this slope. One way to find a is to equate these two slopes.

    Solution: The line is perpendicular to the tangent line, so its slope is $-\frac{1}{2a}$, On the other hand, the segment connecting the two points has slope $\frac{a^2}{a-18}$. Setting these equal to one another gives $18 - a = 2a^3$, which we can solve by observation to get $a = 2$. So the point of the curve closest to $(18, 0)$ is $(2, 4)$.

(d) Build the function $D(x)$ whose value at x is the distance from $(18, 0)$ to the point (x, x^2) on the curve. Then, as we did in class, find the derivative of the function $(D(x)^2$. Evaluate your derivative at each of the points $x = 0, x = 1$, and $x = 2$.

Solution:

(e) Find the distance from $(18, 0)$ to the nearest point on the curve.

Solution: $D((18, 0), (2, 4)) = \sqrt{16^2 + 4^2} = \sqrt{16(17)} = 4\sqrt{17}$.

20. (20 points) Amber Airlines runs chartered flights to Costa Rica. They expect 300 passengers and they charge each passenger \$200. However, if more than 300 persons sign up for the flight, they agree to charge \$0.50 less per ticket for each extra person. For example, if 302 passengers sign up, the airline charges each of the 302 passengers \$199.00.

(a) Note that if $x = 300$, the revenue is $300 \cdot \$200 = \$60000$. Build the revenue function $R(x)$, where x represents the total number of passengers.

Solution: Let x represent the number of passengers beyond 300 that Amber Airlines enlists. Then $R(x) = (x)(200 - 0.25(x - 300))$ if $x \geq 300$.

(b) How many passengers result in the maximum revenue?

Solution: To maximize $R(x)$, find the critical points and the endpoints of the domain. The domain is $[0, \infty)$, and, by the product rule, the derivative is $R'(x) = 1(200 - 0.25(x - 300) + x(-0.25) = 200 - .25x + 75 - .25x = 275 - 0.5x$. So $x = 550$ is the only critical point. Note that $R''(550) = -0.5$, so this means that $x = 550$ is the location of a relative maximum. Since $R(0) = 200 \cdot 300 = 60000$ is the only endpoint, and since R is decreasing to the right of $x = 550$ (why?, $R'(x)$ is negative for $x > 550$), it follows that R has an absolute maximum at $x = 550$.

(c) What is that maximum revenue?

Solution: The maximum revenue is $R(550) = 550 \cdot 137.50 = 75625$.