## Introducing Some Exponential Models

This essay is intended to be read before the third test. The purpose is to familiarize you with some of the applications of exponential functions in daily life.But before we discuss any of these five models, let me try to give you an idea of the type of problems you'll encounter. You are well aware that two pieces of information are required to pin down a linear function. It could be two points or it could be the $y$-intercept and the slope. When we write a line in slope-intercept form $f(x) = mx+b$, we say that we have two *parameters*, $m$ and $b$. In other words, for each pair of numbers $m$ and $b$, there is exactly one line (vertical lines cannot be represented this way, as you know). Now suppose you are given that $f(0) = 4$ and $f'(0) = 3$, Can you use this information to find $f(1)$? Of course you can: $b = 4$ and $m = 3$, so $f(x) = 3x+4$ and $f(1) = 7$. This problem serves to illustrate the types of problems you encounter below. Typically, you have a two or three parameter family (ie set) of functions and two or three pieces of information. Your job is to use the information to find the function and its value at some point.

The first two models we discuss are *exponential growth* and *exponential decay.* We use the letter $Q$ as the function name and $t$ as the independent variable because usually $Q(t)$ refers to the **q**uantity of some item at **t**ime $t$.

1. Exponential Growth. $Q(t) = Q_0e^{kt}$ where both parameters $k$ and $Q_0$ are positive numbers. Notice that continuously compound interest ($A(t) = Pe^{rt}$) is an example, although the letters are different. As we have seen this function has domain $(-\infty, \infty)$ although we usually are not interested in $Q(t)$ for negative values of $t$. Our $Q(t)$s are all increasing functions (note that the derivative is $Q'(t) = Q_0ke^{kt} = kQ(t)$, which is positive for all $t$). Also, they are all concave upwards on their domain. Why?

2. Exponential Decay. $Q(t) = Q_0e^{-kt}$ where again both parameters $k$ and $Q_0$ are positive numbers. Notice that radioactive decay is an example, as we have seen in class. As we have seen this function too has domain $(-\infty, \infty)$ although, once again, we usually are not interested in $Q(t)$ for negative values of $t$. Our $Q(t)$s are all decreasing functions (note that the derivative is $Q'(t) = -Q_0ke^{kt} = -kQ(t)$, which is negative for all $t$). Also, they are all concave upwards on their domain. Why?

In the linear model $f(x) = mx + b$, we use the slope $m$ as a measure of the strength of the function. In the two exponential models above, the number that we use to measure the strength of the function is the doubling time (in 1.) and the half-life (in 2.). You've seen the computation of doubling time in the case of interest on an investment. You might also recall that the doubling times of the *insidious* bacteria is one minute. You might also like to work out why the function $f(t) = 2^t$ is an example of exponential growth. Recall that $2P = Pe^{rt}$ implies that

$rt = \ln(2) \approx 0.693$ Multiply $r$ by 100 to get $r'$ and replace 0.693 with $0.72 \times 100 = 72$ and you get $r' \cdot t = 72$, the famous 'realtors rule of 72'. So you can say that the doubling time for an 8% investment at roughly $72/8 = 9$ years. You've also seen the technique called Carbon-14 dating. C-14 is a radioactive isotope of carbon whose half-life is about 5730 years. You'll see different numbers used for this as well. Thus a piece of tree bark that has exactly half the carbon-14 you'd expect from a living tree is about 5730 years old.

3 Logistic Growth. $Q(t) = \frac{A}{1+Be^{-kt}}$. Once again we insist that all the parameters are positive. Note that $Q(0) = \frac{A}{1+Be^{-k0}} = \frac{A}{1+B}$ is the initial value of $Q$ and $\lim_{t\to\infty} \frac{A}{1+Be^{-kt}} = \frac{A}{1+B\lim_{t\to\infty} e^{-kt}} = A$. This parameter $A$ is sometimes called the *carrying capacity* of the system being modeled. The logistic curve is used in modeling the spread of a disease or the spread of a rumor.

We discussed in class the idea of modeling the spread of an epidemic using the logistic function. Clearly it should start and end spreading slowly. When is it spreading the fastest? The answer seems to be that it's spreading fastest at the time when there is the greatest interaction between infected people and non-infected people. How can we prove that out logistic function has that property? We can show that the largest value of $Q'(t)$ occurs at a $t$ value that satisfies $Be^{-kt} = 1$. In that case we have $1 + Be^{-kt} = 2$ and $Q(t) = A/2$, just what we want. So, let's differentiate $Q$ to get $Q'(t) = A(-1)(1 + Be^{-kt})^{-2}(-kBe^{-kt})$ which we can write as $\frac{ABke^{-kt}}{(1+Be^{-kt})^2}$. So $Q''(t) = -k^2ABe^{-kt}(1+Be^{-kt})^2 - 2(1+Be^{-kt})(-kBe^{-kt})ABke^{-kt}$. We can factor this to get $Q''(t) = (1 + Be^{-kt})(-k^2ABe^{-kt}(1 + Be^{-kt}) + 2AB^2k^2e^{-2kt})$. It follows that $-k^2AB(e^{-kt} - Be^{-2kt}) = 0$ and from this we find that $e^{-kt} = Be^{-2kt}$. Thus $e^{kt} = B$ which is the same as saying $Be^{-kt} = 1$.

Here is an example from an old test. **Spreading of a Rumor.** Three hundred college students attend a lecture of the dean at which she hints that the college will become coed. The rumor spreads according to the logistic curve

$$Q(t) = \frac{3000}{1 + Be^{-kt}},$$

where $t$ is measured in hours.

(a) Compute the parameter $B$.
**Solution:** Note that $Q(0) = 300 = \frac{3000}{1+Be^{-k0}}$, so $1 + B = 10$, and $B = 9$.

(b) How many students attend the college? Hint: the question is not 'how many attended the lecture?'

**Solution:** Take the limit as $t \to \infty$ since eventually everyone knows the rumor. $\lim_{t\to\infty} \frac{3000}{1+9e^{-kt}} = 3000$.

(c) Two hours after the speech, 600 students had heard the rumor. How many students had heard the rumor after four hours?

**Solution:** Solve $Q(2) = 600 = \frac{3000}{1+9e^{-2k}}$ for $k$. We get $9e^{-2k} = 4$ and $k = \frac{\ln 4 - \ln 9}{-2} \approx 0.40546$. Therefore $Q(4) = \frac{3000}{1+9e^{-4k}} = 1080$.

(d) How fast is the rumor spreading after four hours?

**Solution:** We need $Q'(t)$. So write $Q(t) = 3000(1 + 9e^{-kt})^{-1}$ and use the chain rule to get $Q'(t) = \frac{3000 \cdot 9ke^{-kt}}{(1+9e^{-kt})^2}$. Using a calculator, we find that $Q'(4) = \frac{27000ke^{-4k}}{(1+9e^{-4k})^2} \approx \frac{2162.48}{7.716} \approx 280$ students per hour.

(e) After how many hours will 2000 students have heard the rumor?

**Solution:** We need to solve the equation $Q(t) = 2000 = \frac{3000}{1+9e^{-kt}}$ for $t$. We have $1 + 9e^{-kt} = 3/2$, and it follows that $e^{-kt} = 1/18$. We can solve this for $t$ to get $t = \frac{\ln 1 - \ln 18}{-k} = \frac{\ln 18}{k} \approx 7.128$ hours.

4 Learning Curves. The fourth model is called the Learning Curve. $Q(t) = A - Be^{-kt}$. Just as above we again agree that all the parameters are positive numbers. This function is increasing too. Can you prove it? Can you find the horizontal asymptote on the right? Take $\lim_{t\to\infty} Q(t)$ to get $y = A$ as the horizontal asymptote. Here's an example from an old test. **Learning to Tpye.** For a particular person learning to **type**, it was found that the number $N$ of words per minute the person was able to type after $t$ hours of practice, was given by

$$N = N(t) = 100(1 - e^{-0.02t}).$$

Can you recognize this as a learning curve? What is $A$? What is $k$?

(a) After 10 hours of practice how many words per minute could the person type?

**Solution:** $N(10) = 100(1 - e^{-0.02(10)} \approx 18.12$.

(b) What was the **rate** of improvement after 10 hours of practice?

**Solution:** $N'(t) = 100(0 + .02e^{-0.02t}) = 2e^{-0.02t}$ so $N'(10) \approx 1.637$ words per minute per hour of practice time.

(c) What was the **rate** of improvement after 40 hours of practice?

**Solution:** $N'(40) = 2e^{-0.8} \approx 0.898$ words per minute per hour.

5 Newton's Law of Cooling. $Q(t) = T + Ae^{-kt}$, where as usual, the three parameters are all positive. The fifth model is called Newton's Law of Cooling. It is based on the principle that the temperature of a body changes at a rate that is proportional to the difference between the temperature of the body and the temperature of its surroundings. For example, very hot coffee cools at a faster rate (say measured in degrees per minute), than warm coffee. Here's an example from a recent test. The name of the function has been changed to $F$ from $Q$. According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the difference between the object's temperature and that of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

where $T$ is the air temperature, $68°F$, $A$ and $k$ are positive constants, and $t$ is expressed in minutes.

(a) What is the value of $A$?

**Solution:** Since $F(t) = 68 + Ae^{-kt}$, it follows that $F(0) = 68 + A \cdot 1 = 212$ so $A = 144$.

(b) Suppose that after exactly 8 minutes, the temperature of the coffee is $136.6°F$. What is the value of $k$?

**Solution:** Solve $F(t) = 136.6 = 68 + 144e^{-k(8)}$ for $k$ to get $k = 0.09269$.

(c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

**Solution:** Solve the equation $80 = 68 + 144e^{-0.09269t}$ for $t$ to get first $e^{-0.09269t} = 12/144 = 0.083333$, and taking logs of both sides yields $t =$ 26.8 minutes.

(d) How fast is the coffee cooling after 1 minute? How fast is the coffee cooling after 5 minutes? How fast is the coffee cooling after 8 minutes?

**Solution:** Compute $F'(t) = -kAe^{-kt} = -144ke^{-kt}$ at $t = 1$, $t = 5$ and $t = 8$.