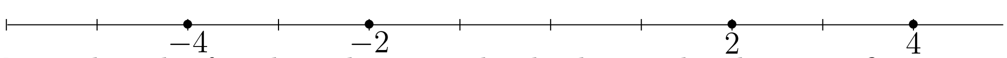
Introducing The Test Interval Technique

This essay is intended to be read before the class has discussed differentiation of elementary functions. One of the fundamental ideas in calculus involves the use of the derivative function in finding the intervals over which a given function is increasing. The method simply requires us to build the sign chart for the derivative, and we can do this for polynomials provided we can write them in factored form. But there are other reasons why we might want to build the sign chart for a function as well. If $f(x)$ is given and we define a new function by $g(x) = \sqrt{f(x)}$, then the domain of g is the set of points for which $f(x) \geq 0$, and if we let $h(x) = \ln(f(x))$, then the domain of h is the set of points for which $f(x) > 0$.

The purpose of this paper is to elaborate on the technique discussed in class for finding the sign chart of a *rational function.* A rational function $r(x)$ is a quotient of two polynomial functions, $p(x)$ and $q(x)$. Of course, if $q(x)$ is the constant function with value 1, then $r(x) = p(x) \div 1 = p(x)$ is a polynomial itself, so all that is said here about rational functions applies to polynomial functions. The sign chart for such a rational function is a depiction of the number line separated into intervals by *branch points.* Plus and minus signs are distributed across the number line depending on the sign of the function at points of the interval. In a nutshell, we first pluck out of the real numbers the places where the function can change sign. For a rational function these points are the zeros of the numerator (aka zeros of the function) and the zeros of the denominator (assuming the rational function is in reduced form, these are the vertical asymptotes). Next, select a test point from each of the intervals determined by the plucking.

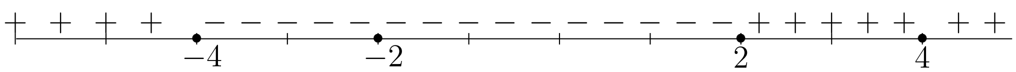
1. Consider the function $p(x) = (x + 4)(x + 2)^2(x - 2)(x - 4)^2$. Note that $p(x)$ is already in *factored form.* The zeros of a polynomial in factored form can be read off without trouble. We have $x = -4, -2, 2$ and 4. The *multiplicities* of -2 and 4 are two. Thus we have four branch points to pluck as shown on the chart below.

   Note that the four branch points divide the number line into five test intervals, $(-\infty, -4), (-4, -2), (-2, 2), (2, 4), (4, \infty)$. Select a *test point* from each interval. Let's take $-5, -3, 0, 3$, and 5.

   To determine the sign of the function at each test point, build a matrix with test points listed down the side and factors listed along the top. In the current case

test point	$(x+4)$	$(x+2)^2$	$(x-2)$	$(x-4)^2$	$p(x)$
-5	$-$	$+$	$-$	$+$	$+$
-3	$+$	$+$	$-$	$+$	$-$
0	$+$	$+$	$-$	$+$	$-$
3	$+$	$+$	$+$	$+$	$+$
5	$+$	$+$	$+$	$+$	$+$



The power of the technique shows up here. It does not matter which point in the interval is selected as the test point. The sign of the function does not change over a test interval. You can see from the sign chart that $p(x)$ changes sign at -4 from positive to negative and at 2 from negative to positive. If the problem we are given is to solve the inequality $p(x) \geq 0$, we could do this easily at this stage. The solution to $p(x) > 0$ is just $(-\infty, -4) \cup (2, 4) \cup (4, \infty)$. There are four zeros of p to add to this set, so we get $(-\infty, -4] \cup \{-2\} \cup [2, \infty)$.

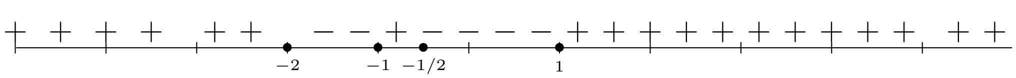
2. Consider the rational function

$$f(x) = \frac{(x^2-4)(2x+1)}{(3x^2-3)(x-2)}.$$

Notice first that f is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$f(x) = \frac{(x-2)(x+2)(2x+1)}{3(x-1)(x+1)(x-2)}.$$

We can cancel the common factors with the understanding that we are (very slightly) enlarging the domain of f: $f(x) = \frac{(x+2)(2x+1)}{3(x-1)(x+1)}$. Next find the branch points. These are the points at which f can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $-2, -1/2, 1, -1$. Again we select test points and find the sign of f at of these points to get the sign chart.



Again suppose that we are solving $f(x) \geq 0$ The solution to $f(x) > 0$ is easy. It is the union of the open intervals with the + signs, $(-\infty, -2) \cup$

$(-1, -1/2) \cup (1, \infty)$. It remains to solve $f(x) = 0$ and attach these solutions to what we have. The zeros of f are -2 and $-1/2$. So the solution to $f(x) \geq 0$ is $(-\infty, -2] \cup (-1, -1/2] \cup (1, \infty)$. Notice that the branch points 1 and -1 are not included since f is not defined at these two points. It has vertical asymptotes at these two places. Technically the value $x = 2$ should not be included in the solution because the function f as originally defined is not defined at $x = 2$. We make this exception repeatedly, however.