Kayles

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Combinatorial Games

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Introduction.

The purpose of this paper is to explore the game of Kayles. Kayles is a combinatorial game in which bowling pins are knocked down at random. A combinatorial game is played with exactly two people. The players must alternate eliminating counters from a limited collection according to some policies. Also, the last player to eliminate a counter is the winner.

Section 1.

Kayles can be categorized as being an impartial game. These games state that either player can make each move. Impartial games are unlike chess because in chess, a player cannot move any of his opponent's pieces. An integer representation is a rule in which describes a correspondence between nonnegative integers and strings of symbols. Some examples of integer representations include binary, decimal, and ternary representations.

A player will soon discover that a binary representation or binary notation knowledge is needed to play Kayles. Binary representation is simply the sum of digital multiples of two. We only need to use the two digits, zero and one, to represent each real number. For example, 1111 means one two-cubed plus one two-squared plus one two plus one $(1*2^3+1*2^2+1*2^1+1*2^0)$. Adding binary numbers is similar to ten-digit addition but with certain stipulations as follows: $(0+1=1), (1+0=1), (0+0=0)$, and $(1+1=10)$. When one is added to one a person would write down zero and carry a one. The result is ten which the binary representation of two.

	0	1
0	0	1
1	1	10

For instance in a game of Kayles with four pins, the game may proceed in various directions. For example, the first player knocks down the middle two. The second player is then faced with two separated pins. Consequently, the second player can only knock down one of the two nonadjacent pins. Thus, the first player wins by knocking down the last pin.

Section 2. Now let's talk about a directed graph or a digraph for a Kayles game of four. A directed graph is a set of vertices corresponding to positions in the game. The lines or edges are the paths a player can follow from one move to the next.

These vertices can be split into two subsets S and U that follow specific properties. The properties include:

1. Zero is in the set S.
2. From every position is U, there is a position of S.
3. Every move from a position in S results in a position in U.

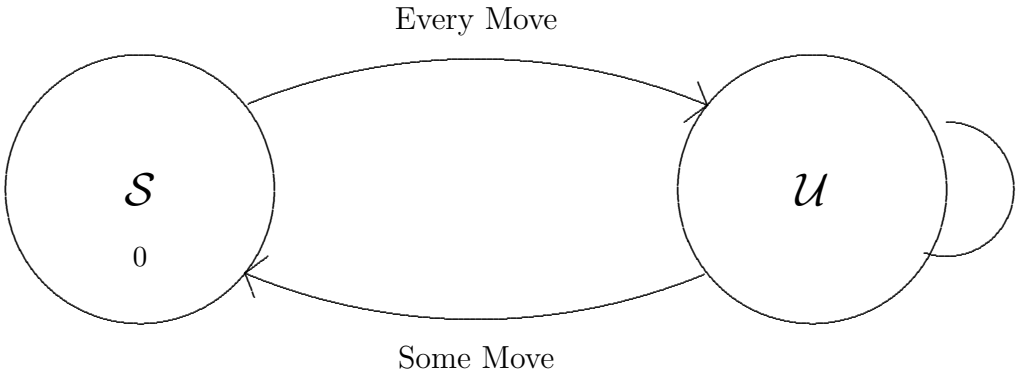

Fig. 2

The players continuously try to achieve the safe (S) positions. Similarly, both players are attempting to avoid unsafe (U) locations. The S-U graph appears as follows for a Kayles game of four:

Instead of classifying positions of a game as safe or unsafe we assign numerical values to each position. All positions in Kayles have Grundy values. Grundy values are simply numbers assign to each position. To find the Grundy value, one must figure out where they can move to from the starting position. After this has been achieved, one must find the minimum excludent or mex and come to the Grundy value. A simply way to define the mex is if T is a limited set of nonnegative integers, the mex (T) denotes the smallest nonnegative integer NOT in T. For example, $Mex(\{0,1,3\}) = 2$, $Mex(\{5,7,9\}) = 0$, and $Mex(\{0,3,4,5,10\}) = 1$.

All positions in safe have a Grundy value of 0. All positive numbers are the Grundy values of the unsafe positions. However, Kayles do not have a definite safe

position, so one must Nim add two positions so they will equal zero. For example in the game with four bowling pins the postion with a zero would fbe 1-1. In Nim addition, one plus one equals zero. The Nim addition table is as follows:

In Nim addition a number plus itself will equal zero unlike binary addition.

The Grundy Values of a game of thirty-five include:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	2	3	1	4	3	2	1	4	2	6	4	1	2	7	1	4	3
19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35		
2	1	4	6	7	4	1	2	8	1	4	7	2	1	8	6	7		

The object of the game is to knock down the last one or two adjacent bowling pins. In order to ensure an absolute win, one should be the first player. If the original number of pins is odd, then the first player should opt to separate the pile into two even piles by removing the middle pin. Likewise, if the number of original pins is even, the first player needs to remove the middle two pins in order to succeed. Positions following the first move must be calculated and examined by analyzing the Grundy values. During the entire game, the winner's goal is be positioned at a safe position with a total Grundy value of zero.