

Wythoff's Game

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Combinatorial Games

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Abstract

In my paper, I will be discussing the rules and winning strategies for Wythoff's combinatorial game. Before a specific game can be explored, the reader must have a general understanding of what a combinatorial game entails. This is a two-person game satisfying certain conditions. These are that the players are removing counters from a finite collection according to some rules and that the last player to remove a counter, according to the previously specified rules, wins. There are two types of games that satisfy these conditions. Those are partisan and impartial games. Since Wythoff's game is an impartial game, those are the games upon which we will focus. In Wythoff's game, the players are confronted with two piles of counters. The players can either take any number of counters from either pile or the same number from both piles. The object of the game is to take the last counter. By coming up with a method of assigning values to each and every move, we create a foolproof winning strategy. These integer representations are rules which describe a correspondence between nonnegative integers and a string of symbols. The system used to denote each position with a number value is called the Grundy value. These values, which will be explained in more depth in my paper, allow us to see the safest positions to move to in the game in order to ensure an absolute win.

Introduction. The purpose of this paper is to explain the rules and winning strategies for Wythoff's combinatorial game. Before a specific game can be explored, the reader must have a general understanding of what a combinatorial game entails. This is a two-person game satisfying certain conditions. Those are that the players remove counters from a finite collection according to some rules and that the last player to remove a counter, in accordance with those rules, wins. There are two types of games that satisfy these conditions, called partisan and impartial. Since Wythoff's game is an impartial game, those are the games upon which we shall be focused. An impartial game is one in which each move can be made by either player. Partisan games, like chess, involve pieces that cannot be moved by the opposite player. In Wythoff's game, the players are confronted with two piles of counters. The players can either take any number of counters from either pile or the same number of counters from both piles. The object of the game is to take the last counter.

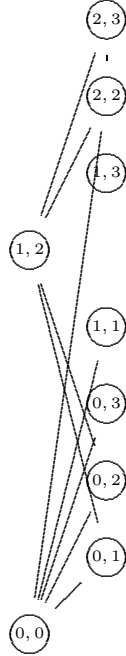


Figure 1

To understand how this, and every combinatorial game, is played perfectly, we must understand where each move takes us in the game. We do this by drawing a directed graph or digraph. This is a set of vertices with a set of directed edges (or moves). Notice in the directed graph in Figure 1 that the game with two and three piles $[W(2,3)]$ could actually be played on the digraph. Every possible move is represented on the graph by a directed edge which, unless otherwise stated, are pointed out or down from the starting position. Each position can be separated into two subsets, S and U , with the following properties:

1. 0 is a member of the set S .
2. From every member of U , there is a move to a member of S .
3. All moves from a position in S result in a member of U .

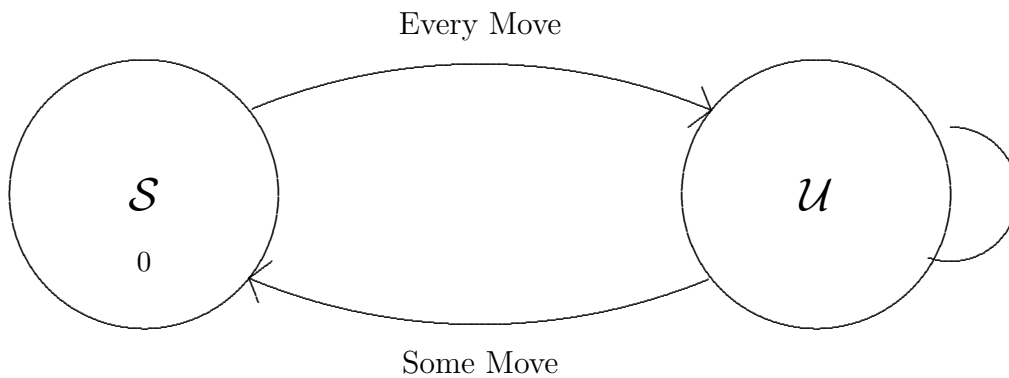


Figure 2

By always moving to a S, or safe, position the player can guarantee a win. Looking back at Fig. 1, we are now able to classify each move as either safe (S) or unsafe (U). Since the position of 0 is the only position that is known, we must start at the bottom and work up the graph. After all the 0, or end, positions are marked with a S, we must look back at the conditions to see where to go next. According to condition 3, it is not possible to move from a safe position to another safe position. Therefore, all the moves that lead to 0 must be unsafe. Since it is impossible to move to the safe position of 0 from (1, 2) it must be a safe position. This type of reasoning can be used to find the safe and unsafe moves in any game.



Figure 3

Since this graph is only on the simple game of $W(2, 3)$, it is obvious that directed graphs can get very complicated, very quickly. So, rather than classifying the positions in a game as safe or unsafe, it is possible to assign a numerical value to each position. These integer representations are rules which describe a correspondence between nonnegative integers and a string of symbols. The system used to denote each position with a number value is called the Grundy value. These values expose the safe positions without a directed graph. It is possible to assign a Grundy value to each position in a game, regardless of which game it is. To calculate the G-value, minimum excludant, or mex, is needed. If T is a finite set of nonnegative integers, $\text{mex}(T)$ denotes the smallest nonnegative integer NOT in T . For example:

$$\text{Mex}(1, 2, 3) = 0$$

$$\text{Mex}(0, 1, 4, 7) = 2$$

$$\text{Mex}(0, 1, 2, 3) = 4 \text{ and}$$

$$\text{Mex}(5) = 0.$$

The Grundy value of every safe position will be 0. Looking back at $W(2, 3)$ in Fig. 1, the Grundy values for $(0,0)$ and $(1,2)$ would be 0. Using the mex function it is easy to calculate the G-value for the other positions as well. Working up the graph, since $(0,1)$ can only move to a 0 position, the mex of that set would be 1. Therefore, the Grundy value is 1. For $(0,2)$, the players are presented with the set 0,1 because it is only possible to move to a $(0,0)$, a position with G-value of 0, and $(0,1)$, which is a position with a G-value of 1. So the Grundy value of $(1,1)$ would be 2, since $mex(0, 1) = 2$.

Instead of making the chart in the form of the directed graph it can be made easier to read by following the ordered pairs. In Figure 5, the two-pile is running down the side and the three-pile is put along the top. It is now possible to see where any move can take you.

If the G-values for each position are deleted, it is easy to see that it is possible to play the game on this board. Starting with the marker at $(2, 3)$, it could be moved up and horizontally as many spaces as would be wanted. It is also possible to move diagonally.

Now that $W(2, 3)$ has been completely analyzed, we are able to concentrate on more complicated games. Starting on a $W(15, 15)$, the calculations for the Grundy values are performed the same way. Though it would be fairly simple to play a game of $W(15, 15)$ because the first player would automatically win, if you have the Grundy values for that game you are able to play any game with piles sizes up to 15.

	0	1	2	3
0				
1				
2				

Figure 4

Remember that the 0 positions are the safe positions in the game. Notice that the 0 entries are $(0, 0)$, $(1, 2)$, $(3, 5)$, $(4, 7)$, etc. Wythoff's difference rule explains the pattern that appears as we continue to study the safe positions. The first number in every pair is the smallest number that hasn't yet appeared. The second number is found by adding one more than the difference of the last ordered pair to the first number. For example, starting after $(0, 0)$, the first number of the ordered pair would be 1 because that's the smallest number that hasn't appeared in the sequence. The difference between $(0, 0)$ is 0, so to find the second number of the next ordered pair we would add 1 to the first number. So we are left with the ordered pair $(1, 2)$.

Now that it is possible for us to play a single game of Wythoff up to $(15, 15)$ perfectly, we can now consider the possibility of playing more than one game at a time. For these operations it is necessary to use the binary representations of the Grundy values. First, we need an accurate understanding of what binary representation means.

The Grundy values completed for Wythoff's Game.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	0	4	5	3	7	8	6	10	11	9	13	14	12	16
2	2	0	1	5	3	4	8	6	7	11	9	10	14	12	13	17
3	3	4	5	6	2	0	1	9	10	12	8	7	15	11	16	18
4	4	5	3	2	7	6	9	0	1	8	13	12	11	16	15	10
5	5	3	4	0	6	8	10	1	2	7	12	14	9	15	17	13
6	6	7	8	1	9	10	3	4	5	13	0	2	16	17	18	12
7	7	8	6	9	0	1	4	5	3	14	15	13	17	2	10	19
8	8	6	7	10	1	2	5	3	4	15	16	17	18	0	9	14
9	9	10	11	12	8	7	13	14	15	16	17	6	19	5	1	0
10	10	11	9	8	13	12	0	15	16	17	14	18	7	6	2	3
11	11	9	10	7	12	14	2	13	17	6	18	15	8	19	20	21
12	12	13	14	15	11	9	16	17	18	19	7	8	10	20	21	22
13	13	14	12	11	16	15	17	2	0	5	6	19	20	9	7	8
14	14	12	13	16	15	17	18	10	9	1	2	20	21	7	11	23
15	15	16	17	18	10	13	12	19	14	0	3	21	22	8	23	20

Figure 5

The numbers we count with (digits 0-9) is in what is called decimal, or base 10, representation. Put simply, that means we have ten digits at our disposal to make any number we desire. In binary, or base 2, representation, we only have access to two digits, 1 and 0. Therefore, to change from decimal representation to binary, we must convert from using ten digits to two. Looking at Fig. 6 we see a different way to think about decimal numbers. The number 495 is broken down into 4 hundreds + 9 tens + 5 ones. Those are all powers of 10. So we could, in fact, say that all decimal numbers are sums of digital multiples of ten. Thus, binary representations would be sums of digital multiples of two. In Figure 7 we see the numbers 1-10 written in binary form.

100	10	1
4	9	5
5	6	1

Figure 6

	8	4	2	1
1				1
2			1	0
3			1	1
4		1	0	0
5		1	0	1
6		1	1	0
7		1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0

Figure 7

Now that we can write the Grundy values for Wythoff's game in binary form, it's possible to play more than one game at once. Take, for example, $W(6, 8)$ and $W(5, 7)$. The Grundy value for the starting position gives you the G-value for the whole game. Therefore, according to the chart in Fig. 5, the Grundy value for $W(6, 8)$ is 5 and $W(5, 7)$ is 1. To find the Grundy value of those two games played together, we must use nim addition. By putting both Grundy values in their binary form, we are able to use a type of addition to get the total Grundy value. Fig. 8 shows both Grundy values in their binary form. In nim addition, $1 + 0$ still equals 1 and $0 + 0$ still equals 0. But remember, in base 2 you can't have any digits higher than 1. So when you're adding $1 + 1$, they must cancel each other out to give you a 0. What if you had to add three ones together? Two of the ones would cancel out, leaving $1 + 0$ which equals 1. If you have an odd number of ones to add together you will always get 1, and if there's an even number of ones it will always equal 0. Back to $W(6, 8) + W(5, 7)$. In Fig. 8, the nim addition has already been worked out and the Grundy value obtained was 100. Looking back at Fig. 7, we see that the binary representation of $100 = 4$ in decimal notation. So the Grundy value of this composite game is 4.

$$\begin{array}{r}
 101 \\
 \oplus 1 \\
 \hline
 100
 \end{array}$$

Figure 8

The purpose of this paper was to inform the reader how to play and win at Wythoff's combinatorial game. This has been done through the discussion of safe and unsafe positions, Grundy values, and nim addition. This exploration has hopefully give you, the reader, enough insight into this game to feel confident enough to play to win.