Games and Representations

SVSM 1999, Combinatorial Games

Harold Reiter

The purpose of this talk (paper) is to explore the relationship between combinatorial games and integer representations. A *combinatorial game* is a two person game satisfying the conditions

1. players alternate removing counters from a finite collection according to some rules and

2. the last player to remove a counter wins.

Later we'll modify this definition to include more games. Strictly speaking, there are two types of games satisfying these conditions, called *partisan* and *impartial* games. We will be concerned here only with impartial games. These are games in which each move can be made by either player (unlike say, chess, where a player cannot move any of his opponent's pieces). An *integer representation* is a rule which describes a correspondence between nonnegative integers and strings of symbols. Binary, decimal, and ternary representations are examples.

Our first example of a combinatorial game is the classic Bouton's Nim. We begin with four piles of counters. For this specific version, we designate piles of sizes $1, 3, 5$, and 7 although we are interested ultimately in playing this game optimally for any number of piles and for piles of any size. The rules permit a player to remove, at his turn, any number of counters from any **one** pile. We denote the initial position by an ordered quadruple $(1, 3, 5, 7)$. The first player will be called Fred and the second Sally. The play of the game can be depicted as a sequence of quadruples, where each move corresponds to an arrow. For example, $(1, 3, 5, 7) \xrightarrow{f} (1, 3, 5, 6) \xrightarrow{s} (1, 3, 4, 6) \xrightarrow{f} (1, 3, 4, 3) \xrightarrow{s} (1, 3, 1, 3) \xrightarrow{f} (1, 3, 1, 0) \xrightarrow{s} (1, 0, 1, 0) \xrightarrow{f} (1, 0, 0, 0) \xrightarrow{s} (0, 0, 0, 0)$. Notice that the game above was won by Sally. The 4-tuples (n_1, n_2, n_3, n_4) are called *positions*, and the set of all positions is denoted by $\mathcal{P}$. The special position $(0, 0, 0, 0)$ is denoted O, as will the terminal position in other games. We'll come back to Bouton's Nim after we learn more about combinatorial games in general and integer representations.

The next step is understanding *place value* numerical representation. We all understand that the number 463.05 is just $4\times10^2+6\times10^1+3\times10^0+0\times10^{-1}+5\times10^{-2}$. In words, four hundreds plus six tens plus three ones plus zero tenths plus five hundredths. That is, 463.05 is a **sum** of **digital multiples** of **powers** of ten. Now suppose we allow ourselves only the powers of 2. This results in what is called *binary* notation or binary representation. In this case we need only use the two digits 0 and 1 to represent (almost uniquely) each real number. For example 1001.01 means 1 two-cubed plus 0 two-squared plus 0 twos plus 1 one plus 0 halves plus 1 fourth, or

9.25. Complete the binary representation sheet (handout 1) provided before reading forward. If you're feeling ambitious, work the arithmetic problems provided also.

Now we turn back to combinatorial games. The simplest of these are called one pile nim. We begin with a single pile of counters and a list of allowable moves. The notation $N(10; 1, 2)$, for example, refers to a game which starts with 10 counters. Players can remove 1 or 2 counters at each turn. Play this game several times with a friend, taking turns as the starting player. Who wins? Try to devise a strategy to win whenever you are the first player.

Next we take another detour, this time to learn about *directed graphs*. A directed graph or *digraph* for short, is a set of *vertices* (here, the positions $0, 1, 2, \ldots, 10$) together with a set of *directed edges* (here, the moves). Here's an example. Notice that the game can be recovered from the graph. In fact, we might as well play the game on the digraph. Unless we say otherwise, all edges are assumed to be downwardly directed. If there is a directed edge from u to v, we say u *communicates* to v. Instead of picking up counters from a pile, we simply move a marker from one numbered position to a lower numbered position along a directed edge.

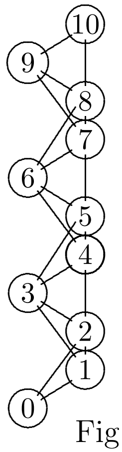

Fig. 1

Notice something very interesting about the digraph. It is that the vertices $\mathcal{P}$ can be split into two subsets $\mathcal{S}$ and $\mathcal{U}$ with the following properties:

1. $0 \in \mathcal{S}$.
2. From every member of $\mathcal{U}$, there is a move to a member of $\mathcal{S}$.
3. All moves from a position in $\mathcal{S}$ result in a member of $\mathcal{U}$.

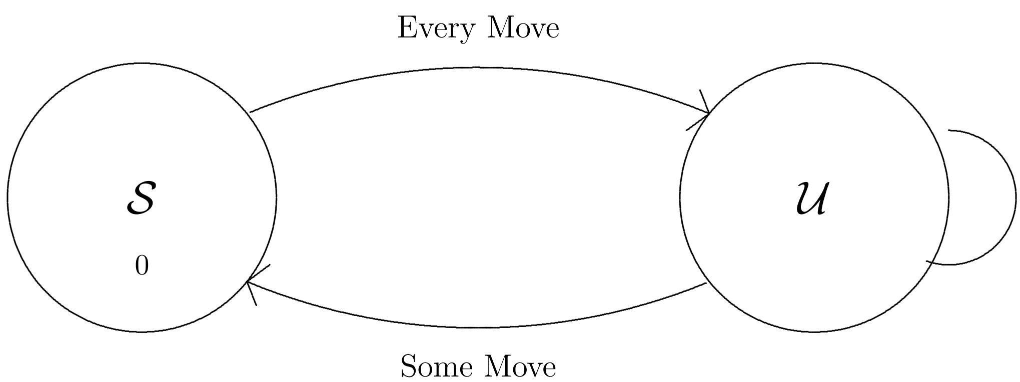

Fig. 2

In our example game, $\mathcal{S}$ $=\{0,3,6,9\}$ and $\mathcal{U}$ $=\{1,2,4,5,7,8,10\}$. How can you win? Simply by moving to a member of $\mathcal{S}$. Then by property 3., your opponent must move to a position in $\mathcal{U}$. Then by property 2., you can move back to a position in $\mathcal{S}$, *etcetera*, and finally, by property 1., you will win because you are the only player who can move to 0. We call the members of $\mathcal{S}$ *safe* positions because they are safe to move **to**, and those of $\mathcal{U}$ *unsafe*. If there is a directed edge from vertex i to vertex j, we call j a *successor* of i.

Let's go back and label each vertex of the digraph with $\mathcal{S}$ or $\mathcal{U}$ using the properties above. Here we abuse the notation, using $\mathcal{S}$ and $\mathcal{U}$ both for sets and for labelling members of those sets. First label vertex 0 with $\mathcal{S}$ because of property 1. Next we see that vertex 1 has all of its successors labeled, so it can be labeled. Since it communicates only to an $\mathcal{S}$ position, we label vertex 1 with $\mathcal{U}$. Now vertex 2 communicates both to a $\mathcal{U}$ position and to an $\mathcal{S}$ position, so how should we mark it? If we mark it $\mathcal{S}$, we violate condition 3., so we must mark it with a $\mathcal{U}$. Now what about vertex 3? We can move from 3 only to 1 and 2, both of which have $\mathcal{U}$ labels. If we label 3 with $\mathcal{U}$, we violate property 2. Therefore we must label vertex 3 with a $\mathcal{S}$. From vertex 2, you'd move to 0 and win rather than to 1 which would let your opponent move to 0 and win. Since you'd win from vertex 2, it is an unsafe position to move to. But 3 is a safe position to move to since your opponent would have to move to 1 or 2 which are both unsafe. We continue the labeling process all the way **up** the graph.

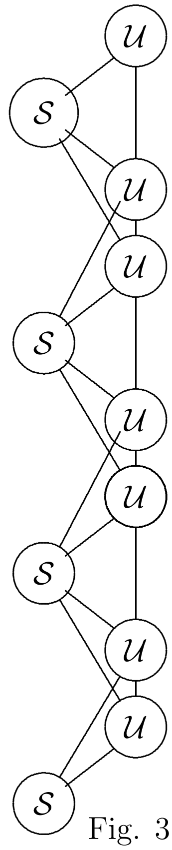

Fig. 3

The general step is this: Suppose all the successor positions of p have been labeled. Then

- if all these positions belong to $\mathcal{U}$, then label p with $\mathcal{S}$, and
- if one of these positions is in $\mathcal{S}$, then give p the label $\mathcal{U}$.

Try the labeling process with the digraph provided in Fig. 4.

Let's try three more games of one pile nim. First, please play with your partner the game $N(20;1,3,5)$; then play $N(20;1,2,5)$; and finally $N(20;1,2,6)$. For each of these games, find the partition of $\mathcal{P}$ into $\mathcal{S}$ and $\mathcal{U}$. Check your work against the tables below:

$N(20;1,3,5)$		$N(20;1,2,5)$		$N(20;1,2,6)$	
position	label	position	label	position	label
20	$\mathcal{S}$	20	$\mathcal{U}$	20	$\mathcal{U}$
19	$\mathcal{U}$	19	$\mathcal{U}$	19	$\mathcal{U}$
18	$\mathcal{S}$	18	$\mathcal{S}$	18	$\mathcal{U}$
17	$\mathcal{U}$	17	$\mathcal{U}$	17	$\mathcal{S}$
16	$\mathcal{S}$	16	$\mathcal{U}$	16	$\mathcal{U}$
15	$\mathcal{U}$	15	$\mathcal{S}$	15	$\mathcal{U}$
14	$\mathcal{S}$	14	$\mathcal{U}$	14	$\mathcal{S}$
13	$\mathcal{U}$	13	$\mathcal{U}$	13	$\mathcal{U}$
12	$\mathcal{S}$	12	$\mathcal{S}$	12	$\mathcal{U}$
11	$\mathcal{U}$	11	$\mathcal{U}$	11	$\mathcal{U}$
10	$\mathcal{S}$	10	$\mathcal{U}$	10	$\mathcal{S}$
9	$\mathcal{U}$	9	$\mathcal{S}$	9	$\mathcal{U}$
8	$\mathcal{S}$	8	$\mathcal{U}$	8	$\mathcal{U}$
7	$\mathcal{U}$	7	$\mathcal{U}$	7	$\mathcal{S}$
6	$\mathcal{S}$	6	$\mathcal{S}$	6	$\mathcal{U}$
5	$\mathcal{U}$	5	$\mathcal{U}$	5	$\mathcal{U}$
4	$\mathcal{S}$	4	$\mathcal{U}$	4	$\mathcal{U}$
3	$\mathcal{U}$	3	$\mathcal{S}$	3	$\mathcal{S}$
2	$\mathcal{S}$	2	$\mathcal{U}$	2	$\mathcal{U}$
1	$\mathcal{U}$	1	$\mathcal{U}$	1	$\mathcal{U}$
0	$\mathcal{S}$	0	$\mathcal{S}$	0	$\mathcal{S}$

Now let's go back to see how to characterize the $\mathcal{S}$ positions of $N(100;1,2)$. You can probably guess that the multiples of 3 play a role here just as they did in $N(10;1,2)$, where $\mathcal{S}$ is the set $\{0,3,6,9\}$. Yes, the positions of $\mathcal{S}$ in $N(100;1,2)$ are the multiples of 3. That is, they are the numbers whose ternary representations have units digit 0. This more complicated characterization of $\mathcal{S}$ positions bridges the gap between one pile nim on one hand and two more complicated games to be introduced later, Bouton's Nim and Whytoff's game on the other. In these more complicated games, the divisibility properties are useless. We are now ready to state our main interest in combinatorial games:

$$\boxed{\text{For each game, find the sets } \mathcal{S} \text{ and } \mathcal{U}\text{, and study their properties.}}$$

This seems to be the right time to fulfill outr promise to enlarge the collection of games of interest. There are lots of games of interest which have nothing to do with counters and which this theory enables us to solve. Some of these are Green Hackenbush, Cram, Kayles and Grundy's game, all of which are studied in the exercises which follow. In fact, however, every finite digraph without cycles is a suitable board for playing a combinatorial game. Consider the one shown here. The top position is the start, and each move is downward along an edge. As usual, the winner is the last player to move. The same $\mathcal{S} - \mathcal{U}$ analysis works here which worked on the one pile nim games earlier. For convenience, we have labeled the vertices a through j. Label the terminal positions with $\mathcal{S}$. In this case there are two, a and j. Then consider any position all of whose successors have been labeled. All successors of b and all successors of i have been labeled, so they can be labeled. Remember that they get label $\mathcal{S}$ if all their successors have $\mathcal{U}$ labels and label $\mathcal{U}$ if there is a successor labeled $\mathcal{S}$. Follow these rules to label all the positions on the digraph of handout 3.

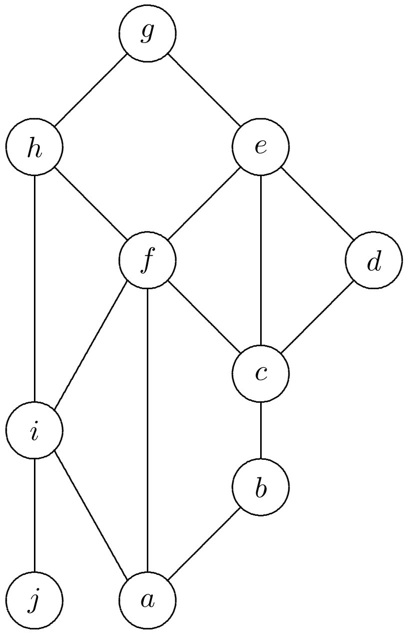

Fig. 4

In fact, however, there is an even more interesting problem which is easier to solve. Rather than classify positions of a game as safe or unsafe, we can assign a numerical value to each position.

Grundy Values of Games

It is possible, no matter what combinatorial game we are playing, to assign to each position a number, called the *Grundy-value* or G-value of the position, in such a way that the positions with G-value 0 are precisely those of $\mathcal{S}$, and all those of $\mathcal{U}$ have positive value. To carry this out, we need a function called **mex**. The **m** stands for **minimum** and the **ex** stands for **excludant**. Here's the definition. If T is a finite set of nonnegative integers, mex(T) denotes the smallest nonnegative integer *NOT* in T. For example,

$$\begin{aligned}
\text{mex}(\{1,2,3\}) &= 0 \\
\text{mex}(\{0,1,4,7\}) &= 2 \\
\text{mex}(\{0,1,2,3\}) &= 4 \text{ and} \\
\text{mex}(\{5\}) &= 0.
\end{aligned}$$

Let's go back to the digraph representing $N(10;1,2)$, assign position 0 the G-value 0 and continue to work upwards, assigning to each position the mex of all the values accessible from that position. In symbols, if p is a position and $S(p) = \{q :$ there is a move from p to $q\}$ is the set of all successors of p, the Grundy-value of the position p, denoted $G(p)$, is given by

$$G(p) = \text{mex}\,(\{G(q) : q \in S(p)\}).$$

Notice that $G(0) = G(3) = G(6) = G(9) = 0$. Compute the G-values of the positions in the digraph game of Fig. 4.

At this stage we consider the relationship between the $\mathcal{S}$,$\mathcal{U}$ classification and the Grundy-value classification. We can prove that the Grundy-value represents a finer classification in the following sense: Each position with G-value 0 is an $\mathcal{S}$ position and each positive G-value position belongs to the set $\mathcal{U}$ of unsafe positions. It is a proof by induction. To be rigorous, we introduce some notation. Let Z and P represent the positions with 0 and positive G-values, respectively. And as before, let $\mathcal{S}$ and $\mathcal{U}$ represent the safe and unsafe positions in the same directed graph game G. Our claim is that $\mathcal{S} = \mathcal{Z}$ and $\mathcal{U} = \mathcal{P}$. First, note that all the terminal positions are both in $\mathcal{S}$ and in Z. Now take a position p all of whose successors have been labeled in both classifications, and assume that for all these positions, the claim holds. If p belongs to $\mathcal{S}$, then it does so because there is no move to an $\mathcal{S}$ position, which means that $G(p) = \text{mex}\,(\{G(q) : q \in S(p)\} = \text{mex}\,(T) = 0$, because $0 \notin T)$. On the other hand, if $p \in \mathcal{U}$, then there is a move from p to a member of $\mathcal{S}$, that is to a position with G-value 0. Hence $G(p) > 0$. This proves that the sets Z and $\mathcal{S}$ coincide and that the sets P and $\mathcal{U}$ coincide.

Next, we learn to play the game $N(10,11)$. This game is played as follows: When it's your turn, select a nonempty pile and take any number of counters from it. Play this game with your partner several times and then try to imagine what the directed graph version looks like. This game is an example of Bouton's Nim, which is played with any number of piles of any sizes in the same way. When it's your turn, select a nonempty pile and take any number of counters from it. The winner is, as usual, the last player to make a move. The directed graph for the game $N(10,11)$ has $(10+1)(11+1) = 132$ vertices and 1265 $(= \frac{11}{2}\cdot 10\cdot 11 + \frac{12}{2}\cdot 11\cdot 10)$ edges, a rather large digraph. To simplify the labeling process, we can draw an 11×12 grid of squares. Imagine the grid located in the plane so that the bottom left corner square has $(0,0)$ at its center and the upper left corner square has $(0,11)$ at its center as shown below.

$(0,11)$										$(10,11)$
$(0,0)$										$(10,0)$

We can play the game on the grid itself instead of using two piles of counters. We start with a marker at the square $(10,11)$. To make a move, slide the marker horizontally to the left or downward. A move to the left corresponds to removing counters from the 10-counter pile, while downward slides correspond to removing counters from the 11-counter pile. Play the game $N(10,11)$ with your partner. Try to anticipate what comes next. First, we'll label on the grid provided the positions with the symbols $\mathcal{S}$ and $\mathcal{U}$, according to weather they are safe or unsafe to move to. One winning strategy for the starting player is clear. You can move from $(10,11)$ to $(10,10)$ and continue to restore the symmetry each time your opponent disturbs it (which he must do on each of his plays). Next, use the other copy of the grid provided to establish the Grundy value of each position. To play Bouton's Nim perfectly, we must understand how to calculate the G-values of these positions quickly. Next we'll view the table of G-values as the results of an operation $\oplus$ on the nonnegative integers. Turn the table 90 degrees so that the upper left corner is the entry $(0,0)$. There is no need to limit ourselves to a 10×11 grid. Let's go for 15×15 for now, but with the understanding that the operation is defined on the entire set of natural numbers $\{0, 1, 2, 3, \ldots\}$. Then the grid looks like:

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

There is another way to think of $N(10, 11)$ which will prove useful later. We can treat it as a pair of single pile games which we agree to play simultaneously. Any finite collection of games can be put together in this way. Such games are called *composite* games. Here's how composite games are played. Suppose we have games G_1 and G_2. Their composite is denoted $G_1 \oplus G_2$. To play $G_1 \oplus G_2$, the first player chooses one of the two component games and makes a move in it. Then the other player chooses one of the games and makes a move in it. This continues until one of the games reaches a terminal position, whereupon both players move in the nonterminated game until it terminates. As usual, the last player making a move wins. Notice that the game $N(10) \oplus N(11)$ is played in just the same way as $N(10, 11)$: t each turn a nonempty pile is picked and take some counters are taken from it. Let's play the composite game $N(12; 1, 2, 5) \oplus N(11; 1, 2, 3)$ to get an idea of how composite games work. To do this, take a pile of 12 counters and another of 11 counters. You're allowed to remove $1, 2$ or 5 from the former pile and $1, 2$ or 3 from the later.

At this stage some astute observers have noticed that we have a symbol $\oplus$ which we are using for two purposes, as a binary operation on games and as a binary operation on nonnegative numbers. We'll have to be careful not to let that confuse us. We will usually be able to tell from context which of the two is meant.

Let's investigate some of the properties of $\oplus$ on the set $Z^+ = \{0, 1, 2, 3, \ldots\}$. We can see that $(Z^+, \oplus)$ has an identity element 0. Its also not hard to see that the

operation is *associative* and *commutative*. Notice that the 0's down the diagonal tell us that each element is its own inverse. Hence the pair $(Z^+, \oplus)$ is what is called an *Abelian group*. There's another important property as well. Each of the subsets $S_i = \{0, 1, 2, \ldots, 2^i - 1\}$ is a *subgroup* of $(Z^+, \oplus)$. What this means is that each of these subsets S_i is closed under the $\oplus$ operation. But how do we compute in $(Z^+, \oplus)$? The answer is this:
Suppose we want to compute $u \oplus v$. Write u and v in binary and 'add' the two binary numbers using the following digit table:

$\oplus$	0	1
0	0	1
1	1	0

Let's translate this into a practical strategy for winning $N(1, 3, 5, 7)$ or any game of Bouton's Nim. For any position, we construct the *binary configuration*. For convenience, we'll go with $N(1, 3, 5, 7)$.

$$\begin{array}{rcccc} 1 & = & & & 1 \\ 3 & = & & 1 & 1 \\ 5 & = & 1 & 0 & 1 \\ 7 & = & \underline{1} & \underline{1} & \underline{1} \\ & & 0 & 0 & 0 = 0 \end{array}$$

In other words, apply the addition table to each column individually. If the result in each column is 0, call the configuration *balanced*, otherwise unbalanced. It turns out that the balanced positions are exactly the $\mathcal{S}$ positions. To prove this we have to see

1. that all terminal positions are balanced,

2. that for each unbalanced position p, there is a move to a balanced position q, and

3. that from any balanced position q, every move results in an unbalanced position.

The first item is clear. There is only one terminal position $O = (0, 0, 0, \ldots 0)$, one 0 for each pile, and its binary configuration is certainly balanced.

Next suppose we have an unbalanced position, say $(u_1, u_2, \ldots, u_k)$. This will be much clearer if we deal with actual values. To be definite, take the position $(5, 10, 15, 20, 25)$ whose binary configuration is given by

$$\begin{array}{rcccccc}
5 & = & & & 1 & 0 & 1 \\
10 & = & & 1 & 0 & 1 & 0 \\
15 & = & & 1 & 1 & 1 & 1 \\
20 & = & 1 & 0 & 1 & 0 & 0 \\
25 & = & \underline{1} & \underline{1} & \underline{0} & \underline{0} & \underline{1} \\
 & & 0 & 1 & 1 & 0 & 1
\end{array}$$

That is, the second, third and fifth columns have an odd number of 1's and the other columns have an even number of 1's. To move from $(5, 10, 15, 20, 25)$ to a balanced position requires finding a way to balance these three unbalanced columns without disturbing the balanced nature of the first and fourth columns. To change the second column to one with an even number of 1's means that we have to remove some counters from one of the three piles which contributes a 1 to that column (why?). That is, we must remove counters from either the 10 pile, the 15 pile or the 25 pile. If we select the 25 pile, we must remove just enough counters to change the 0 that pile contributes in the third column to a 1, leave the 0 contribution in the fourth column, and change the 1 in the fifth column to a 0. We want the pile to contribute 1 0 1 0 0 which is the binary representation of 20. Thus the move $(5, 10, 15, 20, 25) \longrightarrow (5, 10, 15, 20, 20)$ takes the game from a position in $\mathcal{U}$ to a position in $\mathcal{S}$. Finally, we need to be sure that any move from a balanced position results in an unbalanced position. This is easy to see. Each move reduces the size of some pile. The binary representation of that pile changes when the pile is reduced. In particular, some 1 gets changed to a 0. The column in which that 1 appears initially is changed to 0 by the move, and this unbalances that column. We're done. To win at Bouton's Nim, just analyze your initial position. If its balanced, you'll probably lose. If its unbalanced, find a balanced position to move to, and after that continue to move to balanced positions.

Now for the really good news. You can win any composite combinatorial game in the same way. Here's an example that shows how. Suppose we're playing the composite game $N(20; 1, 3, 5) \oplus N(20; 1, 2, 5) \oplus N(20; 1, 2, 6) \oplus W(12, 11) \oplus N(3, 5, 7, 9)$ where $W(12, 11)$ is Whytoff's game with piles of sizes 12 and 11. Whytoff's game is played just like two pile nim except that players can also remove the same number of counters from both piles at a turn. See the homework problem on Whytoff's game. Of course, the game $N(3, 5, 7, 9)$ is itself a composite of the four one pile nim games $N(3), N(5), N(7)$ and $N(9)$. That is, $N(3, 5, 7, 9) = N(3) \oplus N(5) \oplus N(7) \oplus N(9)$. This composite game is played as follows: at each turn a player selects one of the five component games and make a legal move in that game. For example, denoting the initial position by $(20, 20, 20, (12, 11), 3, 5, 7, 9)$, the first player could move to $(20, 20, 20, (9, 8), 3, 5, 7, 9)$, since that corresponds to taking one counter from each of the two Whytoff piles. We know from earlier about the Grundy values of the component games, and we are therefore able construct the binary configuration as follows:

Component Game	Grundy value	Binary rep. of Grundy value
$N(20; 1, 3, 5)$	0	0 0 0 0 0
$N(20; 1, 2, 5)$	2	0 0 0 1 0
$N(20; 1, 2, 6)$	3	0 0 0 1 1
$N(3)$	3	0 0 0 1 1
$N(5)$	5	0 0 1 0 1
$N(7)$	7	0 0 1 1 1
$N(9)$	9	0 1 0 0 1
$W(9, 8)$	15	0 1 1 1 1
Composite	?	0 0 1 1 0

We may conclude that the Grundy value of the composite game is the $\oplus$ sum (or sometimes called the nim sum) of the component games, which is in this case $110_2 = 6$. To restore the balance to the configuration, we need to select a game which contributes a 1 to the third column. Any of the games $N(5)$, $N(7)$, or $W(9, 8)$ will work. Can you find the winning move from $W(9, 8)$?