Using Linear Companion Recursions to Solve Recursive Equations

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Abstract

We give an algorithm for solving recursive equations of the type $a, b, c = \frac{Q(a,b)}{L(a,b)}, d = \frac{Q(b,c)}{L(b,c)}, e = \frac{Q(c,d)}{L(c,d)}, \cdots$ where $Q(a,b)$ and $L(a,b)$ are quadratic and linear functions of a, b. We do this by defining an equivalent linear companion recursion which is easy to solve.

The two variable algorithm also works for any number of variables.

1 Introduction

Problem A3 on the 1979 Putnam Exam stated: Suppose a_n are defined by $a_1 = \alpha, a_2 = \beta, a_{n+2} = a_n a_{n+1}/(2a_n - a_{n+1})$, where α, β are chosen so that $a_{n+1} \neq 2a_n$. For what α, β are infinitely many an integral? There is a trivial solution. By induction we can show that $a_{n+2} = \alpha\beta/((n+1)\alpha - n\beta) = \alpha\beta/(n(\alpha - \beta) + \alpha)$. So we must have $\alpha = \beta$, otherwise the denominator grows without limit. But if $\alpha = \beta$, then all $a_n = \alpha$. So infinitely many a_n are integral if and only if $\alpha = \beta$ is an integer. Alternatively, if we write the recursion $a_{n+2} = \frac{a_n a_{n+1}}{2a_n - a_{n+1}}$ as $\frac{1}{a_{n+2}} = \frac{2a_n - a_{n+1}}{a_n a_{n+1}} = \frac{2}{a_{n+1}} - \frac{1}{a_n}$ and then call $\frac{1}{a_n} = x_n$, we have the trivial linear recursion $x_{n+2} = -x_n + 2x_{n+1}$. In this paper we solve much more complicated recursions in this class of recursions.

This work was motivated by the amazing result of Coxeter-Conway on Wormholes[4]. Although we did not find an elementary explanation of the phenomenon, we were able to create several variations of the wormhole calculations which allowed us to compute sequences of n-tuples all of whose entries are integers. By substituting special integers in our sequences we were able to create sequences of n-tuples all of whose n entries are the same. These sequences of single integers led us naturally to the ideas in this paper. Suppose $Q(\overline{a}, \overline{b}) = A\overline{a}^2 + B\overline{a}\overline{b} + C\overline{b}^2 + D\overline{a} + E\overline{b} + F$ and $L(\overline{a}, \overline{b}) = G\overline{a} + H\overline{b} + I$ are given quadratic and linear functions of variables $\overline{a}, \overline{b}$. Also a and b are given complex numbers. In Section 2 we give an algorithm for solving recursions of the type $a, b, c = \frac{Q(a,b)}{L(a,b)}, d = \frac{Q(b,c)}{L(b,c)}, e = \frac{Q(c,d)}{L(c,d)}, \cdots$ where $Q(a,b)$ and $L(a,b)$ are quadratic and linear functions of a, b. We do this by defining an equivalent linear companion recursion $a, b, c = xb - a + y, d = xc - b + y, e = xd - c + y, f = xe - d + y, \ldots$. In sections 3, 4, and 5 we give four examples of the algorithm. In the fourth example $Q(\overline{a}, \overline{b})$ and $L(\overline{a}, \overline{b})$ are completely general. In sections 7 and 8, we prove that the algorithm works. Of course, some readers will already have figured this out after reading the four examples. The algorithm also works for any number of variables and in section 9 we deal with the 3 variable case, $a, b, c, d = \frac{Q(a,b,c)}{L(a,b,c)}, e = \frac{Q(b,c,d)}{L(b,c,d)}, \ldots$. In our very simple first example of the algorithm we will consider the recursion $a, b, c = \frac{Q(a,b)}{L(a,b)} = \frac{b^2+1}{a}, d = \frac{Q(b,c)}{L(b,c)} = \frac{c^2+1}{b}, \ldots$. Using $a = 1, b = 1$, this recursion gives us the sequence $1, 1, 2, 5, 13, 34, \ldots$, which the algorithm also computes by the linear recursion $a = 1, b = 1, c = 3b - a, d = 3c - b, \ldots$. This example of the odd indexed Fibonacci numbers appeared in [3]. For arbitrary a and b the sequence above can aso be computed by $a, b, c = xb - a, d = xc - b, e = xd - c, \ldots$, where $x = \frac{a^2+b^2+1}{ab}$. Our second application of the algorithm uses $P(\overline{a}, \overline{b}) = \frac{Q(\overline{a},\overline{b})}{L(\overline{a},\overline{b})} = \frac{\overline{b}^2+\theta\overline{b}+\phi\overline{a}+\psi}{\overline{a}+\Delta}$, which generates the recursion $a, b, c = \frac{Q(a,b)}{L(a,b)}, d = \frac{Q(b,c)}{L(b,c)}, \ldots$. In this example, $\theta, \phi, \psi, \Delta$ must be adjusted so that the three conditions of the algorithm can be met. In this paper we have found it more difficult to use the formal notation $x_n = \frac{Q(x_{n-2},x_{n-1})}{L(x_{n-2},x_{n-1})}$.

2 Defining the Algorithm for Solving the Recursions of Section 1

Suppose $Q(\overline{a}, \overline{b}) = A\overline{a}^2 + B\overline{a}\overline{b} + C\overline{b}^2 + D\overline{a} + E\overline{b} + F$ and $L(\overline{a}, \overline{b}) = G\overline{a} + H\overline{b} + I$ are given quadratic and linear functions of variables $\overline{a}, \overline{b}$. Also a and b are given complex numbers. The recursion $a, b, c = \frac{Q(a,b)}{L(a,b)}, d = \frac{Q(b,c)}{L(b,c)}, e = \frac{Q(c,d)}{L(c,d)}, \cdots$ can be solved by the equivalent linear companion recursion $a, b, c = xb - a + y, d = xc - b + y, e = xd - c + y, \ldots$, when the conditions 1,2,3 of the algorithm are met and x and y are specified in the algorithm. As long as a, b, x, y and the coefficients of $Q(\overline{a}, \overline{b}), L(\overline{a}, \overline{b})$ satisfy the three conditions, the algorithm will compute the more complicated recursion using the much simpler linear recursion.

The Conditions

1. First, define $\overline{P}(\overline{a}, \overline{b}, x, y) = Q(\overline{a}, \overline{b}) - L(\overline{a}, \overline{b})(x\overline{b} - \overline{a} + y) = [A\overline{a}^2 + B\overline{a}\overline{b} + C\overline{b}^2 + D\overline{a} + E\overline{b} + F] - [G\overline{a} + H\overline{b} + I][x\overline{b} - \overline{a} + y]$. Condition 1 requires that $* \ \overline{P}(a, b, x, y) = 0$, where a and b are the first two members of the sequence.

2. This condition requires $\overline{P}$ to be symmetric in the variables $\overline{a}$ and $\overline{b}$. That is, $\overline{P}(\overline{a}, \overline{b}, x, y) = \overline{P}(\overline{b}, \overline{a}, x, y)$ In this case, suppose $\overline{P}(\overline{a}, \overline{b}, x, y) = \theta(x, y)\overline{a}^2 + \phi(x, y)\overline{a}\overline{b} + \theta(x, y)\overline{b}^2 + \psi(x, y)\overline{a} + \psi(x, y)\overline{b} + \Delta(x, y)$.

3. Referring to the notation in condition 2, $x = -\frac{\phi(x,y)}{\theta(x,y)}, y = -\frac{\psi(x,y)}{\theta(x,y)}$.

3 Solving the first two Examples.

Here we solve two examples from section 1. We first solve the very simple example $a, b, c = P(a, b), d = P(b, c), e = P(c, d), \cdots$ where $P(a, b) = \frac{Q(a,b)}{L(a,b)} = \frac{b^2+1}{a}$.

From Condition 1 we define $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = Q\left(\overline{a}, \overline{b}\right) - L\left(\overline{a}, \overline{b}\right)\left[x\overline{b} - \overline{a} + y\right] = \left(\overline{b}^2 + 1\right) - \overline{a}\left[x\overline{b} - \overline{a} + y\right] = \overline{a}^2 - x\overline{a}\overline{b} + \overline{b}^2 - \overline{a}y + 1$. From Condition 1, $(*)$ $\overline{P}(a, b, x, y) = a^2 - xab + b^2 - ay + 1 = 0$ must be true.

From Condition 2, we require $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = \overline{P}\left(\overline{b}, \overline{a}, x, y\right)$ and this is true if and only if $y = 0$. Thus, $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = \overline{a}^2 - x\overline{a}\overline{b} + \overline{b}^2 + 1$ and $(*)$ becomes $a^2 - xab + b^2 + 1 = 0$.

From Condition 3, we require $x = \frac{-\phi(x,y)}{\theta(x,y)} = x, y = \frac{-\psi(x,y)}{\theta(x,y)} = 0$, and this condition is automatically satisfied since $x = x$ and $y = 0$.

For arbitrary a, b we need $(*)\ x = \frac{a^2+b^2+1}{ab}$ and the sequences $a, b, c = \frac{b^2+1}{a}, d = \frac{c^2+1}{b}, \cdots$ can be solved by $a, b, c = xb - a, d = xc - b, \cdots$ where $x = \frac{a^2+b^2+1}{ab}$.

We next solve the example $a, b, c = P(a,b), d = P(b,c), e = P(c,d), \cdots$ where $P(a,b) = \frac{Q(a,b)}{L(a,b)} = \frac{b^2+\theta b+\phi a+\psi}{a+\Delta}$ and where we must adjust $\theta, \phi, \psi, \Delta$ so that the 3 conditions of the algorithm can be met.

From Condition 1 we define $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = Q\left(\overline{a}, \overline{b}\right) - L\left(\overline{a}, \overline{b}\right)\left[x\overline{b} - \overline{a} + y\right] = \overline{b}^2 + \theta\overline{b} + \phi\overline{a} + \psi - (\overline{a} + \Delta)\left[x\overline{b} - \overline{a} + y\right] = \overline{a}^2 - x\overline{a}\overline{b} + \overline{b}^2 + (\phi + \Delta - y)\,\overline{a} + (\theta - \Delta x)\,\overline{b} + \psi - \Delta y$. From Condition 1, we require $(*)\ \overline{P}(a, b, x, y) = a^2 - xab + b^2 + (\phi + \Delta - y)\,a + (\theta - \Delta x)\,b + \psi - \Delta y = 0$.

From Condition 2, we require $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = \overline{P}\left(\overline{b}, \overline{a}, x, y\right)$ which is true iff $\phi + \Delta - y = \theta - \Delta x = \psi(x,y)$ where the notation $\psi(x,y)$ is from Condition 2.

From Condition 3, we require $x = \frac{-\phi(x,y)}{\theta(x,y)} = x$ which is automatically true. Note that $\frac{-\phi(x,y)}{\theta(x,y)}$ comes from the notation in Condition 2 of the algorithm and not from the coefficients $\theta, \phi, \psi, \Delta$ that we are using in our example. We also require $y = \frac{-\psi(x,y)}{\theta(x,y)} = y - \phi - \Delta$ and this is equivalent to $\phi + \Delta = 0$. Therefore, the restrictions of the algorithm are $(*)$ and $\phi + \Delta - y = \theta - \Delta x$ and $\phi + \Delta = 0$ which is equivalent to $\phi = -\Delta$ and $y = \Delta x - \theta$ and $(*)\ a^2 - xab + b^2 + (\theta - \Delta x)\,a + (\theta - \Delta x)\,b + \psi - \Delta y = 0$.

Therefore, $P(a,b) = \frac{b^2+\theta b-\Delta a+\psi}{a+\Delta}$ and $y = \Delta x - \theta$ and $(*)\ \overline{P}(a, b, x, y) = a^2 - xab + b^2 + (\theta - \Delta x)\,a + (\theta - \Delta x)\,b + \psi - \Delta y = 0$. However, note that $\psi - \Delta y = \psi + \Delta\theta - \Delta^2 x$.Therefore, if $a, b, \theta, \Delta, \psi$ are given Condition 1 requires $(*)\ x = \frac{a^2+b^2+\theta a+\theta b+\psi+\Delta\theta}{(a+\Delta)(b+\Delta)}$ since $ab + \Delta a + \Delta b + \Delta^2 = (a+\Delta)(b+\Delta)$. Also, of course, $y = \Delta x - \theta$. Therefore, for any fixed $a, b, \theta, \Delta, \psi$ the recursion $a, b, c = P(a,b), d = P(b,c), e = P(c,d) \cdots$ where $P(a,b) = \frac{b^2+\theta b-\Delta a+\psi}{a+\Delta}$ can be solved by $c = xb - a + y, d = xc - b + y, \cdots$ where x is defined above by $(*)$ and $y = \Delta x - \theta$.

In this example, if $a = 1, b = 1, \theta = 1, \psi = 1, \Delta = 0$ we have $x = 5, y = -1$ and $P(a,b) = \frac{b^2+b+1}{a}$ and a sequence $1, 1, 3, 13, 61, \cdots$ which can be solved by $a, b, c = 5b - a - 1, d = 5c - b - 1, e = 5d - c - 1, \cdots$.

In the above calculations if a, b, x, θ, Δ are given then we can adjust ψ so that Condition 1 is met.

Also, of course $\phi = -\Delta$ and $y = \Delta x - \theta$.

4 A Very Important Example

We use the algorithm to solve the recursion $a, b, c = P(a,b), d = P(b,c), e = P(c,d), \cdots$ where $P(a,b) = \frac{Q(a,b)}{L(a,b)} = \frac{\theta b^2+\phi ab+\psi}{a+\Delta b}$. We assume $\Delta \neq 1$.

As always we use the linear function $xb - a + y$ where it turns out that $y = 0$. If $\Delta = 1$ we can also use $xb - a + y$ where $y \neq 0$. We must adjust $\theta, \phi, \psi, \Delta$ so that the 3 conditions of the algorithm can be met.

From Condition 1, we define $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = Q\left(\overline{a}, \overline{b}\right) - L\left(\overline{a}, \overline{b}\right)\left[x\overline{b} - \overline{a} + y\right] = \left(\theta\overline{b}^2 + \phi\overline{a}\overline{b} + \psi\right) - \left(\overline{a} + \Delta\overline{b}\right)\left[x\overline{b} - \overline{a} + y\right] = \overline{a}^2 + (\phi + \Delta - x)\,\overline{a}\overline{b} + (\theta - \Delta x)\,\overline{b}^2 - y\overline{a} - \Delta y\overline{b} + \psi$.

From Condition 1, we require $(*)\,\overline{P}(a,b,x,y)=a^2+(\phi+\Delta-x)\,ab+(\theta-\Delta x)\,b^2-ya-\Delta yb+\psi=0$.

From Condition 2, we require $\overline{P}\left(\overline{a},\overline{b},x,y\right)=\overline{P}\left(\overline{b},\overline{a},x,y\right)$ which is true iff $y=0$ (since we also assume that $\Delta\neq 1$) and $\theta-\Delta x=1$.

Therefore, $(*)$ and $\overline{P}\left(\overline{a},\overline{b},x,y\right)$ become $(*)\,a^2+(\phi+\Delta-x)\,ab+b^2+\psi=0$ and $\overline{P}\left(\overline{a},\overline{b},x,y\right)=\overline{a}^2+(\phi+\Delta-x)\,\overline{a}\overline{b}+\overline{b}^2+\psi$. Also, we have $\theta-\Delta x=1, y=0$.

From Condition 3, we require $x=\frac{-\phi(x,y)}{\theta(x,y)}=x-\phi-\Delta$ and $y=0=\frac{-\psi(x,y)}{\theta(x,y)}=0$. Therefore, $\phi+\Delta=0$.

Therefore, $\Delta=-\phi$ and $\theta=\Delta x+1=1-\phi x$. Also, $y=0$. Therefore, $P(a,b)=\frac{Q(a,b)}{L(a,b)}=\frac{(1-\phi x)b^2+\phi ab+\psi}{a-\phi b}$ and $(*)\,\overline{P}(a,b,x,y)=a^2-xab+b^2+\psi=0$. Also, $y=0$. As long as $(*)\,a^2-xab+b^2+\psi=0$ is satisfied by a,b,x,y then the recursion $a,b,c=P(a,b)\,,d=P(b,c)\,,e=P(c,d)\,,\cdots$ can be solved by the linear companion recursion $a,b,c=xb-a,d=xc-b,e=xd-c,\cdots$.

We note that x appears in $P(a,b)=\frac{Q(a,b)}{L(a,b)}$ and that is why we call this a very important example.

Let us now consider $a=1,b=1,x=3,\phi=-1,\psi=1$. Note that $(*)$ is satisfied and this gives us $P(a,b)=\frac{4b^2-ab+1}{a+b}$ and a sequence 1,1,2,5,13,34,$\cdots$ which is also computed by $a=1,b=1,c=3b-a,d=3c-b,e=3d-c,\cdots$. This is the same sequence generated by $a=1,b=1,c=P(a,b)=\frac{b^2+1}{a},d=P(b,c)=\frac{c^2+1}{b},\cdots$ and $c=3b-a,d=3c-b,e=3d-c,\cdots$ which is the very easy example that we studied first. This is easy to see if we look at it in the right way.

Suppose that $\frac{b^2+1}{a}=3b-a$. Then $3b-a=\frac{b^2+1}{a}=\frac{3b^2-ab}{b}=\frac{\left(b^2+1\right)+\left(3b^2-ab\right)}{a+b}=\frac{4b^2-ab+1}{a+b}$. The reader may need to think about this.

5 The General Case

We now apply the algorithm to solve the recursion computed by $P(a,b)=\frac{Q(a,b)}{L(a,b)}$ where $Q(a,b)\,,L(a,b)$ are completely general quadratics and linear functions of a,b.

Of course, $a,b,c=P(a,b)\,,d=P(b,c)\,,e=P(c,d)\,,\cdots$, is computed by $a,b,c=xb-a+y,d=xc-b+y,e=xd-c+y,\cdots$.

All of our examples are special cases of this most general case. As always the coefficients of $Q(a,b)$, and $L(a,b)$ must be adjusted so that the 3 conditions of the algorithm can be satisfied by $\overline{P}\left(\overline{a},\overline{b},x,y\right),a,b,x,y$.

We define $P(a,b)=\frac{Q(a,b)}{L(a,b)}$ where $Q(a,b)=Aa^2+Bab+Cb^2+Da+Eb+F$ and $L(a,b)=\overline{A}a+Gb+H$. We use $\overline{A}$ for a special reason.

From Condition 1, we define

$$\begin{aligned}
\overline{P}\left(\overline{a},\overline{b},x,y\right) &= Q\left(\overline{a},\overline{b}\right)-L\left(\overline{a},\overline{b}\right)\left[x\overline{b}-\overline{a}+y\right]\\
&= A\overline{a}^2+B\overline{a}\overline{b}+C\overline{b}^2+D\overline{a}+E\overline{b}+F-\left[\overline{A}\overline{a}+G\overline{b}+H\right]\left[x\overline{b}-\overline{a}+y\right]\\
&= \overline{a}^2\left[A+\overline{A}\right]+\overline{a}\overline{b}\left[B-\overline{A}x+G\right]+\overline{b}^2\left[C-Gx\right]\\
&\quad +\overline{a}\left[D-\overline{A}y+H\right]+\overline{b}\left[E-Hx-Gy\right]+F-Hy.
\end{aligned}$$

From Condition 1, we require $(*)\,\overline{P}(a,b,x,y) = a^2\left[A+\overline{A}\right] + ab\left[B-\overline{A}x+G\right] + b^2\left[C-Gx\right] + a\left[D-\overline{A}y+H\right] + b\left[E-Hx-Gy\right] + F - Hy = 0$.

From Condition 2, we require $\overline{P}\left(\overline{a},\overline{b},x,y\right) = \overline{P}\left(\overline{b},\overline{a},x,y\right)$ which is true iff 1. $A+\overline{A} = C-Gx = \theta(x,y)$ and 2. $D-\overline{A}y+H = E-Hx-Gy = \psi(x,y)$ where $\theta(x,y), \psi(x,y)$ is the notation used in Condition 2 of the algorithm.

From Condition 3, we require $x = \frac{-\phi(x,y)}{\theta(x,y)} = \frac{-\left[B-\overline{A}x+G\right]}{A+\overline{A}}$ and $y = \frac{-\psi(x,y)}{\theta(x,y)} = \frac{-\left[D-\overline{A}y+H\right]}{A+\overline{A}}$ which are equivalent to 3. $B+G+Ax=0$ and 4. $D+H+Ay=0$. This gives us restrictions 1, 2, 3, 4.

By combining restrictions 2, 4 we now replace restriction 2 by $2'$. $2'.-\left(A+\overline{A}\right)y = E-Hx-Gy$. Also, restrictions 1, 2, 3, 4 are equivalent to 1, $2'$, 3, 4. Therefore, we have the following 5 conditions that must be satisfied by $A,B,C,D,E,F,\overline{A},G,H,a,b,x,y$.
$(*)\ \overline{P}(a,b,x,y)=0$.

1. $C = A+\overline{A}+Gx$.

$2'$ $E = \left(G-A-\overline{A}\right)y + Hx$.

3. $B = -G-Ax$.

4. $D = -H-Ay$.

Since F does not appear in Conditions 1, $2'$, 3, 4 we can easily adjust F so that the condition $(*)\,\overline{P}(a,b,x,y)=0$ is met.

Since we have 5 conditions to be met we see that in general 8 of the 13 variables $A,B,C,D,E,F,\overline{A},G,H,a,b,x,y$ can be independent.

We also note that this general solution checks out with the 3 examples that we have studied.

We now summarize what we know.

If $A,B,C,D,E,F,\overline{A},G,H,a,b,x,y$ satisfies the above 5 conditions $(*)$ 1, $2'$, 3, 4 then the recursion $a,b,c = P(a,b), d = P(b,c), e = P(c,d), \cdots$ where $P(a,b) = \frac{Q(a,b)}{L(a,b)} = \frac{Aa^2+Bab+Cb^2+DA+Eb+F}{\overline{A}a+Gb+H}$ can be solved by the linear companion recursion $a,b,c = xb-a+y, d = xc-b+y, e = xd-c+y, \cdots$.

As a specific example, let $a=1, b=1, A=1, \overline{A}=0, B=-5, C=5, D=0, E=0, F=2, G=1, H=0, x=4, y=0$.

We see that this is compatible with the above 5 conditions $(*)\,1,2',3,4$. Therefore, $P(a,b) = \frac{a^2-5ab+5b^2+2}{b}$ and the recursion, $a=1, b=1, c=P(a,b)=P(1,1)=3, d=P(b,c)=P(1,3)=11, e=P(c,d)=P(3,11)=41, \cdots$ can be solved by the linear recursion $a=1, b=1, c=xb-a+y=4b-a=3, d=4c-b=11, e=4d-c=41, \cdots$.

Of course, we also note that we can reverse the sequence by computing $3=P(41,11), 1=P(11,3), 1=P(3,1)$. As a problem for the reader we let the reader solve the recursion $P(a,b) = \frac{-ab+(1+x)b^2+F}{a+b}, a=b=1$, by the liner companion recursion $xb-a+y$ where y must be adjusted so that the first 3 terms of the sequence $a,b,c,d,\cdots = 1,1,\frac{x+F}{2},\cdots$ are correctly computed by the linear recursion $a,b,c,d,\cdots = 1,1,xb-a+y = x-1+y,\cdots$. This is the same as adjusting y so that the condition $\overline{P}(a,b,x,y)=0$ is met.

6 A Problem

Suppose a, b, x, y are given and the sequence $a, b, c, d, e, \cdots$ is generated by $a, b, c = xb - a + y, d = xc - b + y, e = xd - c + y, \cdots$.

We wish to generate this exact same sequence by using a non-linear function $P(a, b)$. This problem can be solved in many different ways if we use Section 5.

An easy solution is to use the second example that we solved in Section 3. After we solved this example, we had $P(a, b) = \frac{b^2 + \theta b - \Delta a + \psi}{a + \Delta}$ where $y = \Delta x - \theta$ and where $(*)\, \overline{P}(a, b, x, y) = a^2 - xab + b^2 + (\theta - \Delta x) a + (\theta - \Delta x) b + \psi - \Delta y = 0$. For convenience, let $\Delta = 0$. Then $\theta = -y$ and $P(a, b) = \frac{b^2 - yb + \psi}{a}$ and $(*)\, \overline{P}(a, b, x, y) = a^2 - xab + b^2 - ya - yb + \psi = 0$.

For given a, b, x, y if ψ is adjusted so that $(*)$ is satisfied then the linear recursion $a, b, c = xb - a + y, d = xc - b + y, e = xd - c + y, \cdots$ is also generated by the non-linear recursion $a, b, c = P(a, b), d = P(b, c), e = P(c, d), \cdots$ where $P(a, b) = \frac{b^2 - yb + \psi}{a}$.

As an example if $a = 1, b = 1, x = 4, y = 0$ then $\psi = 2$ satisfies $(*)$ and the sequence $1, 1, 3, 11, 41, \cdots$ generated by $a = 1, b = 1, c = 4b - a, d = 4c - b, e = 4d - c, \cdots$ is also generated by $a = 1, b = 1, c = P(a, b), d = P(b, c), e = P(c, d), \cdots$ where $P(a, b) = \frac{b^2 + 2}{a}, \cdots$.

The reader might like to solve the following problem. Define $P(a, b) = \frac{b^2 - 3b + \psi}{a}$. For given a, b find ψ so that the sequence $a, b, c = P(a, b), d = P(b, c), e = P(c, d), \cdots$ repeats itself in blocks of 6.

Hints: Consider using $a, b, x = 1, y = 3$.

7 A Result Needed in Proving the Algorithm

The following technique is the core of the algorithm.

Suppose $\overline{P}(a, b) = \theta a^2 + \phi ab + \theta b^2 + \psi a + \psi b + \Delta = 0$ is a symmetric quadratic equation in the two variables a, b.

If $\overline{P}(a, b) = \overline{P}(b, a) = 0$ then $\overline{P}(\overline{a}, b) = \overline{P}(b, \overline{a}) = 0$ and $\overline{P}\left(a, \overline{b}\right) = \overline{P}\left(\overline{b}, a\right) = 0$ where $\overline{a} = -\frac{\phi b}{\theta} - a - \frac{\psi}{\theta} = xb - a + y$ and $\overline{b} = -\frac{\phi a}{\theta} - b - \frac{\psi}{\theta} = xa - b + y$.

Note that $x = \frac{-\phi}{\theta}, y = \frac{-\psi}{\theta}$ depend only on the coefficients θ, ϕ, ψ of the symmetric quadratic $\overline{P}(a, b)$. x, y are completely independent of a, b. Also, by symmetry x, y are the same in both $\overline{a} = xb - a + y$ and $\overline{b} = xa - b + y$.

We also note that $\overline{P}(a, b) = \overline{P}(b, a) = 0$ and $\overline{P}(\overline{a}, b) = \overline{P}(b, \overline{a}) = 0$ are equivalent.

In other words, $\overline{P}(a, b) = \overline{P}(b, a) = 0$ if and only if $\overline{P}(\overline{a}, b) = \overline{P}(b, \overline{a}) = 0$.

Also, $\overline{P}(a, b) = \overline{P}(b, a) = 0$ and $\overline{P}\left(a, \overline{b}\right) = \overline{P}\left(\overline{b}, a\right)$ are equivalent.

This fact is very important in the algorithm.

Also, we note that (1) $\overline{P}(a, b) = 0$ is true if and only if (2)$\overline{P}(b, c) = 0$ is true where $c = xb - a + y$.

Also, (2) $\overline{P}(b, c) = 0$ is true iff (3) $\overline{P}(c, d) = 0$ is true where $d = xc - b + y$.

Also, (3) $\overline{P}(c, d) = 0$ is true iff (4) $\overline{P}(d, e) = 0$ is true where $e = xd - c + y$, etc.

Thus, starting with a, b we have created a sequence $a, b, c, d, e \cdots$ that is defined by $c = xb - a + y, d = xc - b + y, e = xd - c + y \cdots$ such that $\overline{P}(a, b) = 0, \overline{P}(b, c) = 0, \overline{P}(c, d) = 0, \overline{P}(d, e) = 0, \cdots$.

This falling domino type effect forms the very core of the algorithm that we gave in Section 2.

Also, we note that in the backward direction we have the sequence e, d, c, b, a where $c = xd - e + y, b = xc - d + y, a = xb - c + y$. Also, we can keep extending the sequence in the backward direction $e, d, c, b, a, \cdots$.

Finally we note that everything in this paper works for any number of variables a, b or a, b, c or a, b, c, d, etc.

We prove this in Section 9 for 3 variables.

8 Reviewing and Proving the Algorithm

Suppose $P(a,b) = \frac{Q(a,b)}{L(a,b)}$ where $Q(a,b)$ is a quadratic and $L(a,b)$ is a linear function of the variables a, b. In the algorithm it is convenient to think of a, b as being variables that have always been assigned to have specific values. In other words a, b are constants that can be adjusted so that certain conditions are met. For fixed a, b we can create the recursive sequence $a, b, c = P(a,b) = \frac{Q(a,b)}{L(a,b)}, d = P(b,c) = \frac{Q(b,c)}{L(b,c)}, e = P(c,d) = \frac{Q(c,d)}{L(c,d)}, \cdots$.

We now review and prove an algorithm that can solve this type of recursion by creating an equivalent linear companion recursion $a, b, c = xb - a + y, d = xc - b + y, e = xd - c + y, \cdots$.

After first defining the function $\overline{P}\left(\overline{a}, \overline{b}, x, y\right)$ we need to adjust a, b, x, y and the coefficients of $P(a,b) = \frac{Q(a,b)}{L(a,b)}$ so that the following 3 conditions on $\overline{P}\left(\overline{a}, \overline{b}, x, y\right)$ are met. It does not matter how these adjustments are made as long as the 3 conditions on $\overline{P}\left(\overline{a}, \overline{b}, x, y\right)$ are met. For clarity we are using the notation $\overline{P}\left(\overline{a}, \overline{b}, x, y\right)$ instead of $\overline{P}(a, b, x, y)$.

1. For variables $\overline{a}, \overline{b}$ we define $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = Q\left(\overline{a}, \overline{b}\right) - L\left(\overline{a}, \overline{b}\right)\left[x\overline{b} - \overline{a} + y\right]$. Now $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = 0$ is true if and only if $P\left(\overline{a}, \overline{b}\right) = \frac{Q\left(\overline{a}, \overline{b}\right)}{L\left(\overline{a}, \overline{b}\right)} = x\overline{b} - \overline{a} + y$. We emphasize that any $\left(\overline{a}, \overline{b}\right)$ that satisfies $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = 0$ will automatically satisfy $P\left(\overline{a}, \overline{b}\right) = x\overline{b} - \overline{a} + y$. Condition 1 requires $(*)\, \overline{P}(a, b, x, y) = 0$ to be true.

2. We need $\overline{P}\left(\overline{a}, \overline{b}, x, y\right)$ to be a symmetric quadratic in the variables $\overline{a}, \overline{b}$. That is, we need $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = \overline{P}\left(\overline{b}, \overline{a}, x, y\right) = \theta(x,y)\,\overline{a}^2 + \phi(x,y)\,\overline{a}\overline{b} + \theta(x,y)\,\overline{b}^2 + \psi(x,y)\,\overline{a} + \psi(x,y)\,\overline{b} + \Delta(x,y)$ to be true.

3. We need $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = 0$ to be true if and only if $\overline{P}\left(\overline{b}, \overline{c}, x, y\right) = 0$ is true where $\overline{c} = x\overline{b} - \overline{a} + y$. From section 7 we know that this will be true if we require the following to be true. Therefore Condition 3 requires that $\overline{c} = x\overline{b} - \overline{a} + y = \frac{-\phi(x,y)}{\theta(x,y)}\overline{b} - \overline{a} - \frac{\psi(x,y)}{\theta(x,y)}$ which is true if and only if $x = \frac{-\phi(x,y)}{\theta(x,y)}, y = \frac{-\psi(x,y)}{\theta(x,y)}$.

From Condition 1 of the algorithm we know that $(*)\, \overline{P}(a, b, x, y) = 0$. Also, $\overline{P}\left(\overline{a}, \overline{b}, x, y\right) = 0$ is true if and only if $P\left(\overline{a}, \overline{b}\right) = x\overline{b} - \overline{a} + y$ is true. From Condition 3 and the falling domino effect stated in section 7 we know that (a) $\overline{P}(a, b, x, y) = 0$ if and only if $\overline{P}(b, c, x, y) = 0$ where $c = xb - a + y$. Also, (b) $\overline{P}(b, c, x, y) = 0$ iff $\overline{P}(c, d, x, y) = 0$ where $d = xc - b + y$.

Also, (c) $\overline{P}(c,d,x,y)=0$ iff $\overline{P}(d,e,x,y)=0$ where $e=xd-c+y$, etc. Also, from Condition 1, (a) $\overline{P}(a,b,x,y)=0$ iff $P(a,b)=c=xb-a+y$. Also, (b) $\overline{P}(b,c,x,y)=0$ iff $P(b,c)=d=xc-b+y$. Also, (c) $\overline{P}(c,d,x,y)=0$ iff $P(c,d)=e=xd-c+y$, etc.

Therefore, we see that the recursion $a,b,c=P(a,b), d=P(b,c), e=P(c,d), \cdots$ can also be computed by the equivalent linear companion recursion $a,b,c=xb-a+y, d=xc-b+y, e=xd-c+y, \cdots$.

Also, we can reverse the sequence $a,b,c,d,e,\cdots$ and compute $c=P(e,d), b=P(d,c), a=P(c,b)$.

In the last section 9 we extend the same algorithm to 3 variables a,b,c. Also the same algorithm can be extended to any number of variables. As always we need to emphasize again that as long as a,b,x,y and the coefficients of $P(a,b)=\frac{Q(a,b)}{L(a,b)}$ satisfy the 3 conditions of the algorithm then the algorithm will work for these $a,b,x,y,P(a,b)=\frac{Q(a,b)}{L(a,b)}$.

In other words the algorithm will correctly compute the recursion $a,b,c=P(a,b), d=P(b,c), e=P(c,d), \cdots$ by the linear recursion $a,b,c=xb-a+y, d=xc-b+y, \cdots$. It does not matter how we adjust a,b,x,y, and the coefficients of $P(a,b)=\frac{Q(a,b)}{L(a,b)}$ to meet these 3 conditions.

9 Generalizing the Algorithm for 3 Variables.

The algorithm of Section 2 can be generalized to any number of variables. In this section we generalize the algorithm to 3 variables a,b,c.

Suppose $P(a,b,c)=\frac{Q(a,b,c)}{L(a,b.c)}$ where $Q(a,b,c)$ is a quadratic and $L(a,b,c)$ is a linear function of the 3 variables a,b,c.

As always it is convenient to think of a,b,c as being variables that have been assigned to have specific values. In other words, a,b,c are constants that can be adjusted so that certain conditions are met.

For fixed a,b,c we can create the recursive sequence $a,b,c,d=P(a,b,c), e=P(b,c,d), f=P(c,d,e), \cdots$.

We now show that the algorithm of Section 2 can be modified to solve this type of recursion by creating an equivalent linear companion recursion $a,b,c,d=xb+xc-a+y,\ e=xc+xd-b+y,\ f=xd+xe-c+y,\cdots$.

Of course, as always the coefficients of $Q(a,b,c)$ and $L(a,b,c)$ may need to be adjusted so that the 3 conditions of the algorithm can be met. After defining the function $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)$ we need to adjust a,b,c,x,y and the coefficients of $P(a,b,c)=\frac{Q(a,b,c)}{L(a,b.c)}$ so that the following 3 conditions on $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)$ are met. As always it does not matter how these adjustments are made as long as the 3 conditions on $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)$ are met.

For clarity we are using the notation $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)$ instead of $\overline{P}(a,b,c,x,y)$.

1. For variables $\overline{a},\overline{b},\overline{c}$, we define $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)=Q\left(\overline{a},\overline{b},\overline{c}\right)-L\left(\overline{a},\overline{b},\overline{c}\right)\left[x\overline{b}+x\overline{c}-\overline{a}+y\right]$. Now $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)=0$ is true if and only if $P\left(\overline{a},\overline{b},\overline{c}\right)=\frac{Q\left(\overline{a},\overline{b},\overline{c}\right)}{L\left(\overline{a},\overline{b},\overline{c}\right)}=x\overline{b}+x\overline{c}-\overline{a}+y$.

   We emphasize that any $\left(\overline{a},\overline{b},\overline{c}\right)$ that satisfies $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)=0$ will automatically satisfy $P\left(\overline{a},\overline{b},\overline{c}\right)=x\overline{b}+x\overline{c}-\overline{a}+y$. Condition 1 requires $(*)\,\overline{P}(a,b,c,x,y)=0$ to be true.

2. We need $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)$ to be a symmetric quadratic in the variables $\overline{a},\overline{b},\overline{c}$. That is, we need $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)=\overline{P}\left(\overline{a},\overline{c},\overline{b},x.y\right)=\cdots=\overline{P}\left(\overline{c},\overline{b},\overline{a},x,y\right)=\theta\left(x,y\right)\overline{a}^2+\theta\left(x,y\right)\overline{b}^2+\theta\left(x,y\right)\overline{c}^2+\phi\left(x,y\right)\overline{a}\overline{b}+\phi\left(x,y\right)\overline{ac}+\phi\left(x,y\right)\overline{b}\overline{c}+\psi\left(x,y\right)\overline{a}+\psi\left(x,y\right)\overline{b}+\psi\left(x,y\right)\overline{c}+\Delta\left(x,y\right)$ to be true.

3. We need $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)=0$ to be true if and only if $\overline{P}\left(\overline{b},\overline{c},\overline{d},x,y\right)=0$ is true where $\overline{d}=x\overline{b}+x\overline{c}-\overline{a}+y$. From section 7 we know that this is true if the following is true. Therefore, Condition 3 requires that $\overline{d}=x\overline{b}+x\overline{c}-\overline{a}+y=-\frac{\phi(x,y)}{\theta(x,y)}\overline{b}-\frac{\phi(x,y)}{\theta(x,y)}\overline{c}-\overline{a}-\frac{\psi(x,y)}{\theta(x,y)}$ which is true if and only if $x=\frac{-\phi(x,y)}{\theta(x,y)},y=\frac{-\psi(x,y)}{\theta(x,y)}$.

From Condition 1 of the algorithm we know that $(*)\,\overline{P}\left(a,b,c,x,y\right)=0$. Also, $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)=0$ is true if and only if $P\left(\overline{a},\overline{b},\overline{c}\right)=x\overline{b}+x\overline{c}-\overline{a}+y$.

From Condition 3 and the falling domino effect stated in section 7 we know that (a) $\overline{P}\left(a,b,c,x,y\right)=0$ if and only if $\overline{P}\left(b,c,d,x,y\right)=0$ where $d=xb+xc-a+y$. Also, (b) $\overline{P}\left(b,c,d,x,y\right)=0$ iff $\overline{P}\left(c,d,e,x,y\right)=0$ where $e=xc+xd-b+y$. Also, (c) $\overline{P}\left(c,d,e,x,y\right)=0$ iff $\overline{P}\left(d,e,f,x,y\right)=0$ where $f=xd+xe-c+y$, etc. Also, from Condition 1, (a) $\overline{P}\left(a,b,c,x,y\right)=0$ iff $P\left(a,b,c\right)=d=xb+xc-a+y$. Also, (b) $\overline{P}\left(b,c,d,x,y\right)=0$ iff $P\left(b,c,d\right)=e=xc+xd-b+y$. Also, (c) $\overline{P}\left(c,d,e,x,y\right)=0$ iff $P\left(c,d,e\right)=f=xd+xe-c+y$, etc.

Therefore, we see that the recursion $a,b,c,d=P\left(a,b,c\right),e=P\left(b,c,d\right),f=P\left(c,d,e\right),\cdots$ can also be computed by the equivalent linear companion recursion $a,b,c,d=xb+xc-a+y,e=xc+xd-b+y,f=xd+xe-c+y,\cdots$.

As a project the reader might like to adjust a,b,c,x,y and the coefficients of $P\left(a,b,c\right)=\frac{Q(a,b,c)}{L(a,b,c)}$ so that the 3 conditions of the algorithm are met for the case where $Q\left(a,b,c\right)$ and $L\left(a,b,c\right)$ are completely general. In the most general case we have $Q\left(a,b,c\right)=Aa^2+Bb^2+\overline{C}c^2+Dab+Eac+Fbc+Ga+Hb+Ic+J$, $L\left(a,b,c\right)=Ka+Lb+Mc+N$. Of course, we can extend the algorithm to any number of variables. As a very simple example, we now solve $P\left(a,b,c\right)=\frac{Q(a,b,c)}{L(a,b,c)}=\frac{b^2+\theta bc+c^2+\phi}{a}$. Given a,b,c,θ it turns out that we must adjust ϕ,x,y so that the 3 conditions of the algorithm are met. It turns out that the algorithm will work for any arbitrary a,b,c,θ but ϕ will need to be adjusted.

From Condition 1 we define $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)=Q\left(\overline{a},\overline{b},\overline{c}\right)-L\left(\overline{a},\overline{b},\overline{c}\right)\left[x\overline{b}+x\overline{c}-\overline{a}+y\right]=\overline{b}^2+\theta\overline{b}\overline{c}+\overline{c}^2+\phi-\overline{a}\left[x\overline{b}+x\overline{c}-\overline{a}+y\right]=\overline{a}^2+\overline{b}^2+\overline{c}^2+\theta\overline{b}\overline{c}-x\overline{a}\overline{b}-x\overline{ac}-\overline{a}y+\phi$.

Now $(*)\,\overline{P}\left(a,b,c,x,y\right)=0$ must be true. Therefore, $(*)\,a^2+b^2+c^2+\theta bc-xab-xac-ay+\phi=0$ must be true.

From Condition 2, we require $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)$ to be a symmetric quadratic in the variables $\overline{a},\overline{b},\overline{c}$.

Therefore, $x=-\theta,y=0$ and $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)=\overline{a}^2+\overline{b}^2+\overline{c}^2+\theta\overline{b}\overline{c}+\theta\overline{a}\overline{b}+\theta\overline{ac}+\phi$. Also, $(*)$ becomes $(*)\,a^2+b^2+c^2+\theta bc+\theta ab+\theta ac+\phi=0$.

From Condition 3, we require $x=\frac{-\phi(x,y)}{\theta(x,y)}=-\theta,y=\frac{-\psi(x,y)}{\theta(x,y)}=0$ where $\theta\left(x,y\right),\phi\left(x,y\right),\psi\left(x,y\right)$ is the notation used in Condition 2.

Of course, we already know that $x=-\theta,y=0$ is true from the symmetry condition on $\overline{P}\left(\overline{a},\overline{b},\overline{c},x,y\right)$ from Condition 2.

Therefore, if a, b, c, θ are given and $\phi = -a^2 - b^2 - c^2 - \theta bc - \theta ab - \theta ac$ and $x = -\theta, y = 0$ then the recursion $a, b, c, d = P(a, b, c) \equiv \frac{b^2+\theta bc+c^2+\phi}{a}, e = P(b, c, d), f = P(c, d, e), \cdots$ can be solved by the equivalent linear companion recursion, $a, b, c, d = xb + xc - a, e = xc + xd - b, f = xd + xe - c, \cdots$.

As a specific example, let $a = b = c = 1$, and $\theta = -2$. Then $x = -\theta = 2, y = 0$ and $\phi = -1 - 1 - 1 + 2 + 2 + 2 = 3$.

The recursion $a = 1, b = 1, c = 1, d = P(a, b, c) = \frac{(b-c)^2+3}{a} = 3, e = \frac{(c-d)^2+3}{b} = 7, f = \frac{(d-e)^2+3}{c} = 19, g = \frac{(e-f)^2+3}{d} = 49, \cdots$ is solved by $a = 1, b = 1, c = 1, d = 2b + 2c - a = 3, e = 2c + 2d - b = 7, f = 2d + 2e - c = 19, g = 2e + 2f - d = 49, \cdots$.

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