Quarto without the Twist

Arthur Holshouser
3600 Bullard St.
Charlotte, NC 28208, USA

Harold Reiter
Department of Mathematics,
University of North Carolina Charlotte,
Charlotte, NC 28223, USA
hbreiter@email.uncc.edu

October 29, 2003

Abstract

The game Quarto, created by Blaise Muller and published by Gigamic, was one of the five Mensa Games of the Year in 1993 and has received other international awards. It is played on a 4 by 4 board of squares with 16 pieces. These pieces show all combinations of size (short or tall), shade (light or dark), solidity (shell or filled), and shape (circle or square). Two players take turns placing the pieces one at a time on the board. The object is to get four pieces in a straight line (row, column or main diagonal) with the same characteristic - all tall for example. Only one piece can go in a cell and, once placed, the pieces stay put. In the regular version, each player chooses his own pieces. In the twist version, the opponent chooses the piece that the moving player must place. In this paper we give a second player winning strategy for the regular version of the game. In the appendix we will also include a slightly flawed strategy that allows the second player to beat a naive player 99.99% of the time and also never lose. We thought the reader would like to see this since this slightly flawed strategy gives the first player at least an outside chance of forcing a tie. Luc Goossens, http://ssel.vub.ac.be/Members/LucGoossens/ has shown that the twist version of Quarto always ends in a draw, given best play [1].

Notation

The most obvious way to represent the 16 pieces is as 16 distinct 4-bit strings of 0's and 1's. However, in this paper, we choose to represent the 16 pieces as $\{a,b,c,d\}$, $\{a,b,c,d'\}$, $\{a,b,c'd\}$, $\{a,b,c'd'\}$, $\{a.b'c,d\}$, $\{a,b',c,d'\}$, $\cdots \{a',b',c'd\}$, $\{a',b',c',d'\}$. The symbols a,b,c and d can be chosen arbitrarily in any order from the four sets {tall, short}, {round, square}, {shell, solid}, {light, dark}, where we choose one characteristic from each of the four sets.

Also, a',b',c',d' is the opposite of a,b,c,d, respectively. We will call a and a', b and b', c and c', d and d' complements. As an example, $a =$ round, $a' =$ square, $b =$ light, $b' =$ dark, $c =$ solid, $c' =$ shell, $d =$ tall, $d' =$ short.

Second Player Winning Strategy

Let us first match the 16 cells of the board 1-1 as follows,

1	5	5	1
2	6	6	2
3	7	7	3
4	8	8	4

.

Let us call this 1-1 onto function f. Thus, if x is the upper left cell, $f(x)$ is the upper right cell. Of course, $f(x) \neq x$ and $f(f(x)) = x$. Thus f is an involution without fixed points.

Let us now match the 16 pieces 1-1 as follows. If we call this 1-1 onto function θ, then $\theta(x) \neq x$ and $\theta(\theta(x)) = x$. Thus θ is also an involution without fixed points.

$$\left| \begin{matrix} abcd \\ abcd' \end{matrix} \right| \begin{matrix} abc'd \\ abc'd' \end{matrix} \left| \begin{matrix} ab'cd \\ ab'cd' \end{matrix} \right| \begin{matrix} ab'c'd \\ ab'c'd' \end{matrix} \left| \begin{matrix} a'bcd \\ a'bcd' \end{matrix} \right| \begin{matrix} a'bc'd \\ a'bc'd' \end{matrix} \left| \begin{matrix} a'b'cd \\ a'b'cd' \end{matrix} \right| \begin{matrix} a'b'c'd \\ a'b'c'd' \end{matrix} \right| .$$

As an example, $\theta(\{a,b',c',d\}) = \{a,b',c',d'\}$. Of course, a,b,c,d can be chosen $8 \cdot 6 \cdot 4 \cdot 2 = 384$ different ways.

We observe that f maps rows into the same row, columns into different columns and main diagonals into the other main diagonal. Note in general that if $x \in \{a,a'\}, y \in \{b,b'\}, z \in \{c,c'\}, v \in \{d,d'\}$, then $\theta(\{x,y,z,v\}) = \{x,y,z,v'\}$.

The second player has a winning strategy. Each time after the first player has made a move, the second player will do the following:

1. First, the second player studies the board to see if he has a winning move anywhere. If he has a winning move somewhere on the board, he makes this winning move and wins the game by having four pieces in a straight line with a common characteristic.

2. If the second player does not have a winning move, he places the piece $\theta(x)$ in the cell $f(y)$, where placing piece x in the cell y was the preceding move by the first player. For example, if the preceding move by the first player was $ab'cd$ in the upper left cell, the second player will place $ab'cd'$ in the upper right cell.

Proof. Let us first show that this strategy will either result in a draw or a win for the second player.

To show this suppose $2n$ moves have been made where $1 \leq n \leq 7$, and the game has not been won. We claim that on move $2n+1$ the first player cannot win. Therefore, suppose the first player has a winning move on move $2n+1$. Now by the definition of f, it is obvious that he cannot have a winning move along any of the 4 rows. This means he must have a winning move along either a column or a diagonal. The reasoning is the same for either the column or the diagonal. So as an example, suppose on move $2n+1$ the first player can place a piece in the first (i.e., left) column and win by having 4 pieces in the first column with a common characteristic, x, x, x, x for example, where $x \in \{a, a', b, b', c, c', d, d'\}$. Now by the definition of f and θ, this means we have the following two cases.

Case 1. If $x \in \{a, a', b, b', c, c'\}$, then the fourth column before move $2n+1$ has x, x, x in it. Therefore, before the second player made move $2n$, either the left column had x, x, x in it or the right (i.e., fourth) column had x, x, x in it. This means the second player, who would have had an x available to him, could have won on move $2n$. Note that since an x is available to the first player on move $2n+1$, an x was also available to the second player on move $2n$.

Case 2. If $x \in \{d, d'\}$, then the fourth column before move $2n+1$ has x', x', x' in it. Therefore, before the second player made move $2n$, either the left column would have x, x, x in it or the right column would have had x', x', x' in it. We observe that the characteristic $\{d, d'\}$, up to move $2n$, had been played in conjugate pairs. For example, if the first player places down d, the second player will follow this by placing down d' and vice-versa.

Since $1 \leq n \leq 7$, this means that the second player before making move $2n$ would have had both an x and an x' available to him, and this means the second player could have won on move $2n$.

Let us now show that the second player always wins by using this strategy.

To do this let us assume that 16 moves have been made and the second player has not won. Of course, this means that the game ended in a draw. We will show that this assumption leads to a contradiction. Since the game is over and nobody won, this means that no four pieces in a straight line have a common characteristic, and in particular no four pieces in a straight line have either the common characteristic d or the common characteristic d'. In the following it is convenient to ignore the characteristic d and d'. This means that we will imagine that the characteristic d or d' has been omitted from each of the 16 pieces.

Let us now focus on the first fourteen moves of this game. By symmetry we have the following two cases, where the squares with the $\times$ represent the last two moves of the game.

$\times$			$\times$

(1)

	$\times$	$\times$	

(2)

We observe that in rows 1, 2, 3 of (1), (2), we must have the following

x, y, z	x', y', z'	x', y', z'	x, y, z

, where $\{x, x'\} = \{a, a'\}, \{y, y'\} = \{b, b'\}, \{z, z'\} = \{c, c'\}$. This is because no row, column or main diagonal can have four pieces with a common characteristic and we are ignoring $\{d, d'\}$. Of course, x, y, z can change as we go from rows 1 to 2 to 3.

By symmetry, there is no loss of generality in assuming that the first row is

abc	$a'b'c'$	$a'b'c'$	abc

. There are only six possibilities for the second row, and we now go through these six possibilities. We also must consider the two cases (1), (2) above for each of these six possibilities. Note that the 16 pieces are now reduced to the following since the pair d, d' has been omitted.

abc	abc'	$ab'c$	$ab'c'$	$a'bc$	$a'bc'$	$a'b'c$	$a'b'c'$
abc	abc'	$ab'c$	$ab'c'$	$a'bc$	$a'bc'$	$a'b'c$	$a'b'c'$

.

<u>Case 1</u>.

abc	$a'b'c'$	$a'b'c'$	abc
abc'	$a'b'c$	$a'b'c$	abc'
C^*			C^*
$\times$			$\times$

The C^*, C^* means that a contradiction has been found. The reasoning for this goes as follows. There is no way that C^* can have both of the characteristics a' and b'. Suppose as an example that C^* has the characteristic a. Also, by symmetry, suppose the left column

abc
abc'
C^*
$\times$

has been filled in the 14 step move sequence before the right column is filled. The second player could easily have filled in the $\times$'d square of the first column and won by having four a's in the first column. He had such an a available of course. From this reasoning we can see that not only is it true that no row, column or main diagonal has 4 pieces with a common characteristic, but also no column or main diagonal that contains an $\times$'d square can have 3 pieces with a common characteristic.

Case 2.

abc	$a'b'c'$	$a'b'c'$	abc
abc'	$a'b'c$	$a'b'c$	abc'
	C^*	C^*	
	$\times$	$\times$	

The contradictions C^*, C^* are in columns 2 and 3.

Case 3.

abc	$a'b'c'$	$a'b'c'$	abc
$a'b'c$	abc'	abc'	$a'b'c$
	C^*	C^*	
$\times$			$\times$

The contradictions C^*, C^* are in the main diagonals, and the reasoning is similar to case 1.

Case 4.

abc	$a'b'c'$	$a'b'c'$	abc
$a'b'c$	abc'	abc'	$a'b'c$
	$\times$	$\times$	

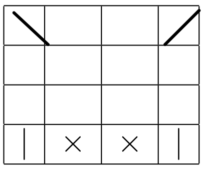
We consider cases (a) and (b),

abc	$a'b'c'$	$a'b'c'$	abc
$a'b'c$	abc'	abc'	$a'b'c$
$ab'c'$	$a'bc$	$a'bc$	$ab'c'$
$ab'c$	$\times$	$\times$	$ab'c$

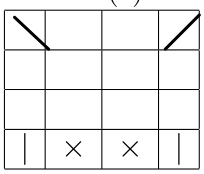
(a)

abc	$a'b'c'$	$a'b'c'$	abc
$a'b'c$	abc'	abc'	$a'b'c$
$a'bc'$	$ab'c$	$ab'c$	$a'bc'$
$a'bc$	$\times$	$\times$	$a'bc$

(b)



(a')



(b')

The two middle cells of the third row from the top of (a), (b) must be either $\{a'bc\}, \{a'bc\}$ or $\{ab'c\}, \{ab'c\}$. Once these two cells are filled, the remaining four cells in (a),(b) are easily filled. However, (a),(b) cannot be realized as the drawings (a'), (b') show. The $\boxed{|}$ in the left and right columns, $\begin{array}{|c|}\hline \\ \hline - \\ \hline \\ \hline | \\ \hline \end{array}$, mean that in the 14 step move sequence, $\boxed{|}$ must be the last cells of their respective columns filled. Also, the $\boxed{\diagdown}$ and $\boxed{\diagup}$ in the main two diagonals means that these cells $\boxed{\diagdown}$ and $\boxed{\diagup}$ must be the last cells in their respective diagonals to be filled. However, the requirements $\boxed{|}$, $\boxed{|}$, $\boxed{\diagdown}$ and $\boxed{\diagup}$ in the 14 step move sequence contradict one another.

The reasoning in the remaining eight cases is similar and we omit the explanations.

<u>Case 5</u>.

abc	$a'b'c'$	$a'b'c'$	abc
$ab'c$	$a'bc'$	$a'bc'$	$ab'c$
C^*			C^*
$\times$			$\times$

The contradictions C^*, C^* are in the first and last columns. The reason is that both a' and c' cannot appear in C^*, C^*.

<u>Case 6</u>.

abc	$a'b'c'$	$a'b'c'$	abc
$ab'c$	$a'bc'$	$a'bc'$	$ab'c$
	C^*	C^*	
	$\times$	$\times$	

The contradictions C^*, C^* are in the second and third columns.

<u>Case 7</u>.

abc	$a'b'c'$	$a'b'c'$	abc
$a'bc'$	$ab'c$	$ab'c$	$a'bc'$
	C^*	C^*	
$\times$			$\times$

The contradictions C^*, C^* are in the main diagonals.

<u>Case 8</u>.

abc	$a'b'c'$	$a'b'c'$	abc
$a'bc'$	$ab'c$	$ab'c$	$a'bc'$
	$\times$	$\times$	

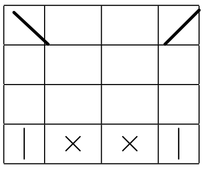
Again we consider two cases, (a) and (b).

abc	$a'b'c'$	$a'b'c'$	abc
$a'bc'$	$ab'c$	$ab'c$	$a'bc'$
$a'b'c$	abc'	abc'	$a'b'c$
$a'bc$	$\times$	$\times$	$a'bc$

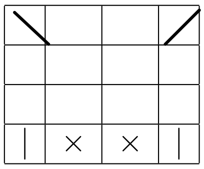
(a)

abc	$a'b'c'$	$a'b'c'$	abc
$a'bc'$	$ab'c$	$ab'c$	$a'bc'$
$ab'c'$	$a'bc$	$a'bc$	$ab'c'$
abc'	$\times$	$\times$	abc'

(b)



(a')



(b')

The two middle cells in the third row from the top of $(a), (b)$ must be either $\{abc'\}, \{abc'\}$ or $\{a'bc\}, \{a'bc\}$. Once these two cells are filled, the remaining 4 cells in $(a), (b)$ are easily filled.

<u>Case 9</u>.

abc	$a'b'c'$	$a'b'c'$	abc
$ab'c'$	$a'bc$	$a'bc$	$ab'c'$
	C^*	C^*	
$\times$			$\times$

The contradictions C^*, C^* are in the main diagonals.

<u>Case 10</u>.

abc	$a'b'c'$	$a'b'c'$	abc
$ab'c'$	$a'bc$	$a'bc$	$ab'c'$
	$\times$	$\times$	

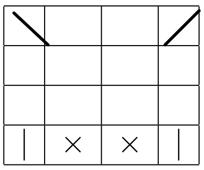
Again we consider two cases.

abc	$a'b'c'$	$a'b'c'$	abc
$ab'c'$	$a'bc$	$a'bc$	$ab'c'$
$a'b'c$	abc'	abc'	$a'b'c$
$ab'c$	$\times$	$\times$	$ab'c$

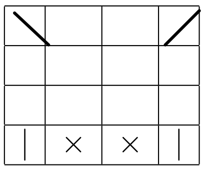
(a)

abc	$a'b'c'$	$a'b'c'$	abc
$ab'c'$	$a'bc$	$a'bc$	$ab'c'$
$a'bc'$	$ab'c$	$ab'c$	$a'bc'$
abc'	$\times$	$\times$	abc'

(b)



(a')



(b')

The two middle cells in the third row from the top of $(a),(b)$ must be either $\{abc'\},\{abc'\}$ or $\{ab'c\},\{ab'c\}$. The other cells of $(a),(b)$ are then filled in.

Case 11.

abc	$a'b'c'$	$a'b'c'$	abc
$a'bc$	$ab'c'$	$ab'c'$	$a'bc$
C^*			C^*
$\times$			$\times$

.

The contradictions C^*, C^* are in the first and fourth columns.

Case 12.

abc	$a'b'c'$	$a'b'c'$	abc
$a'bc$	$ab'c'$	$ab'c'$	$a'bc$
	C^*	C^*	
	$\times$	$\times$	

.

The contradictions C^*, C^* are in the second and third columns. $\square$

Appendix

A strategy allowing the second player to beat a naive opponent 99.99% of the time and also never lose.

The following strategy is much more entertaining since the first player has an outside chance of forcing a draw. The first player, however, can never win. We will omit the proof. The second player uses the same strategy as in section 3 except he now uses the following involutions for f and θ.

1	2	3	4
5	6	7	8
2	1	4	3
6	5	8	7

$abcd$	$abcd'$	$abc'd$	$abc'd'$	$ab'cd$	$ab'cd'$	$ab'c'd$	$ab'c'd'$
$a'b'c'd'$	$a'b'c'd$	$a'b'cd'$	$a'b'cd$	$a'bc'd'$	$a'bc'd$	$a'bcd'$	$a'bcd$

References

[1] Luc Goossens, http://ssel.vub.ac.be/Members/LucGoossens/quarto/quartotext.htm

[2] Richard K. Guy, *Fair Game*, 2nd. ed., COMAP, New York, 1989.