TITLE:

Don’t Fence Me In! Counting and Geometrical Thinking with the Lattice Octagon Problem

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Attendees at the 2009 NCTM Annual Meeting in Washington, DC, were presented with the *Daily Puzzle Challenge*, a collection of problems that could be solved for prizes. The following problem, which was written by one of the authors of this article, appeared in the *Daily Puzzle Challenge* for Thursday, April 23:

> *Create a polygon with eight sides such that all adjacent sides are perpendicular and the side lengths are 1, 2, 3, 4, 5, 6, 7, and 8 units.*
> (NCTM 2009, p. 3)

The scoring for the problem indicated that points would be awarded based on the area of the polygon that was created. In other words, puzzle solvers were asked to find the largest rectangular lattice octagon using side lengths ranging from 1 to 8 units. Since its release, this puzzle has become known as the *Lattice Octagon Problem*.

A *lattice polygon* is a polygon whose vertices are points of a regularly spaced array. Therefore, a *rectangular lattice octagon* is a lattice polygon where each of the eight sides is perpendicular to its adjacent sides. One example of a rectangular lattice octagon is shown in **Figure 1**.

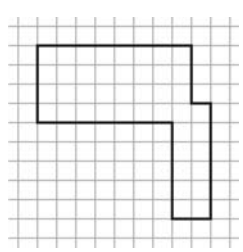

Figure 1. A rectangular lattice octagon.

The lattice octagon in Figure 1 satisfies the requirement that each side length from 1 to 8 units must be used. However, it has an area of only 43 square units, which is significantly less than the maximum possible area.

This article provides a framework for solving this problem and other problems that involve lattice polygons. In addition, it offers a number of suggestions for how to use this problem in the classroom, as well as offering related problems that could be investigated.

Investigating the Lattice Octagon Problem in the Classroom

To begin a classroom investigation, the following wording may provide a better statement of the original problem:

> *What is the largest octagon in which adjacent sides are perpendicular and the side lengths are 1, 2, 3, 4, 5, 6, 7, and 8 units (not necessarily in order)?*

Students can obviously explore this problem using pencil and graph paper. A slightly better alternative is a geoboard, which allows students to easily determine the area. The online geoboard available at http://standards.nctm.org/document/eexamples/Chap4/4.2/standalone.htm uses a 10 × 9 grid, so students can calculate the area directly or they can calculate the area of the unused squares and subtract from 90. A rectangular lattice octagon created using this online manipulative is shown in **Figure 2**. The area of the octagon in Figure 2 is 52 square units.

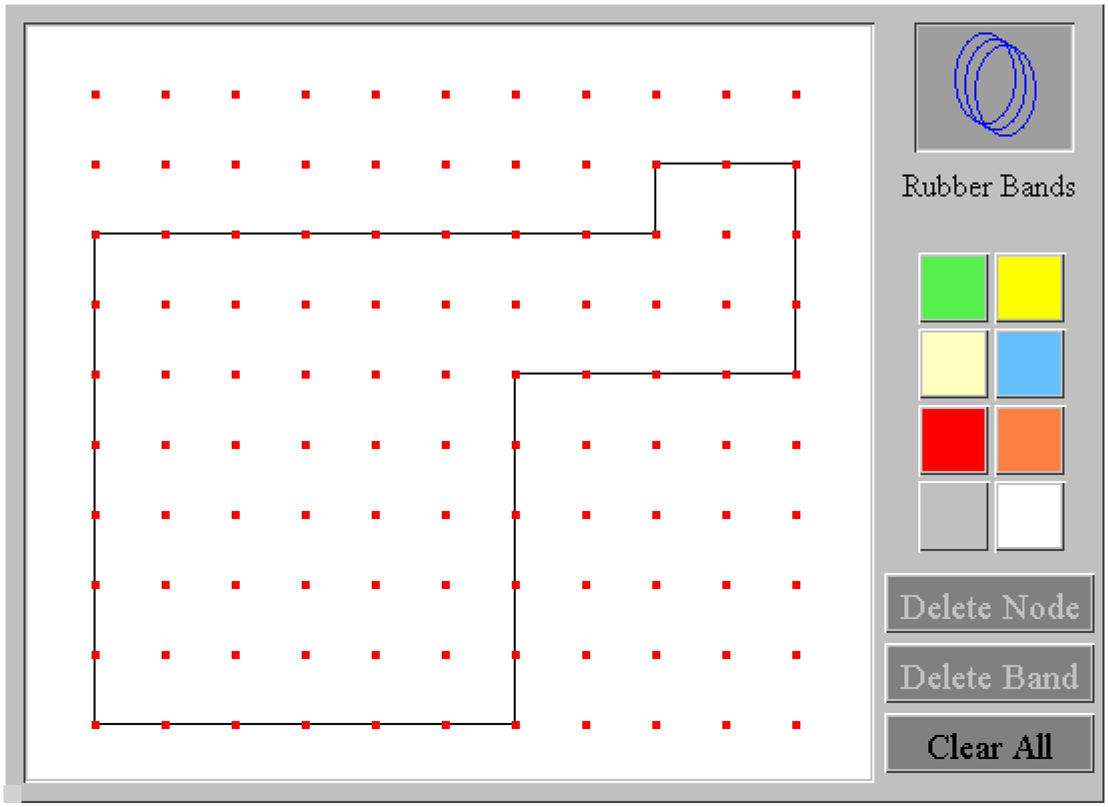

Figure 2. A lattice octagon created using the interactive geoboard applet from the NCTM E-Examples.

Better still is to provide students with a large piece of felt paper on which a grid of squares has been drawn; in addition, provide eight thin strips of felt that are 1 through 8 units in length. The benefit of using felt is that the strips will stick to the board but are not permanently affixed, so they can be rearranged easily during investigation. The set of materials shown in **Figure 3** was made from two pieces of felt

(99¢ each) and a Sharpie® permanent marker ($1.49) purchased at a local craft store. A lattice with lines 3/4” apart was drawn on the lighter sheet of felt, and the strips (1/4” wide) were cut from a darker sheet. Once the materials were purchased, the entire set was constructed in about 10 minutes. To save preparation time, though, you could have students prepare the boards and strips at the beginning of class. Alternatively, students could prepare the materials as homework the night before the lesson, and they could even begin the exploration of the problem as part of their homework. Alternatively, we recommend creating several sets using felt, and then offering the Lattice Octagon Problem on Monday as a problem of the week. The sets that have been prepared can be left in the back of the classroom for student investigation, and students can explore the problem when time permits. Tell students that they can use the felt sets to investigate the problem throughout the week, that they should keep notes on any discoveries they make, and that the full solution will be discussed on Friday.

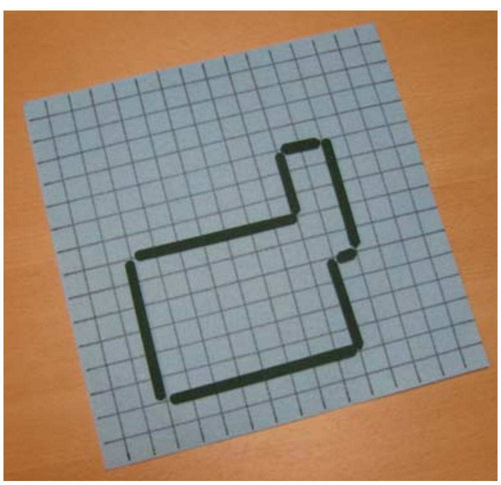

Figure 3. Grid of squares and strips made from felt.

Using the manipulatives, students will likely make two important discoveries.

First, students may discover that lattice octagons can be constructed by first creating a rectangular frame with the eight segments, selecting two corners, and then pushing each of the adjacent segments toward the center. One example of this transformation is shown in **Figure 4**, in which two different lattice octagons are generated from the same frame. Second, students will realize that most octagons can be rearranged easily to create a different octagon. Consider the two lattice octagons created in **Figure 4**. In a sense, these lattice octagons are *complementary.* Both begin with a rectangle using the same arrangement of the eight segments around the perimeter, but different segments are pushed toward the center to create a lattice octagon. Consequently, the resulting octagons have different shapes and therefore different areas.

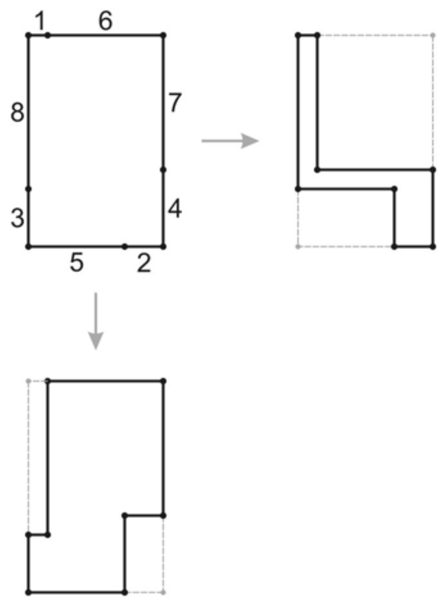

Figure 4. Complementary lattice octagons made from the same rectangular frame.

Students might also realize that octagons with a hollow interior are possible. These octagons are generated from rectangular frames in which one side consists of three segments and the opposite side consists of just one segment. Such is the case in **Figure 5**, where the left side of the frame consists of three segments with lengths of 1, 2, and 3 units, while the right side of the frame consists of just one segment with a length of 6 units. With such a frame, it is only possible to create an octagon only by pushing segments that lie on the same side of the frame. From there, the jutting rectangle can be pushed toward the center to create an octagon with a hollow interior.

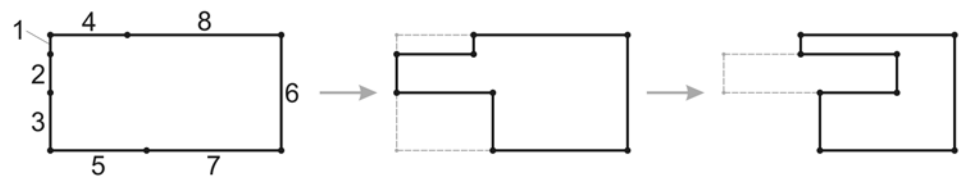

Figure 5. Another set of complementary lattice octagons, the second of which has a hollow interior.

In the course of exploring lattice octagons, students will make many discoveries. With each discovery, they come one step closer to solving the original problem. In addition, they are also building a foundation for investigating several extensions.

Related Problems

One criterion of a good math problem is that it leads to other problems worthy of investigation. The following are some questions that will likely arise when students begin to explore the Lattice Octagon Problem:

1. Can the sides of the octagon be placed in order from end to end? That is, can the sides be placed so that the 1-unit side is adjacent to the 2-unit side, which is adjacent to the 3-unit side, which is adjacent to the 4-unit side, and so on? For convenience, designate this arrangement as *S* = {1, 2, 3, 4, 5, 6, 7, 8}.
2. What is the greatest area among all lattice octagons?
3. What is the least area among all lattice octagons?
4. How many non-congruent lattice octagons are there?

5. Is it possible to build a lattice decagon using side lengths from 1 to 10?

All of these questions allow for rich explorations in a high school mathematics class. As it turns out, the fourth problem in the list is the key to answering all five questions. Therefore, the solution to Problem 4 will be presented in detail, and the answers to most of the other questions will be revealed within the solution to Problem 4.

To solve Problem 4, first note that the sum of the interior angles of any polygon is $(n - 2) \times 180°$, so the sum of the interior angles of an octagon is $(8 - 2) \times 180° = 1080°$. The measure of each interior angle of a rectangular lattice octagon is either 90° or 270°. Let x be the number of 90° angles, and let y be the number of 270° angles. Then it must be the case that $90x + 270y = 1080$. But we also know that $x + y = 8$, because there are eight angles in an octagon. This system of equations implies that $x = 6$ and $y = 2$, so it follows that the measures of exactly two angles must be 270°, while the measures of the other six must be 90°. This realization leads to several possible cases. The two 270° angles can be:

a) adjacent,

b) separated by one 90° angle,

c) separated by two 90° angles, or

d) separated by three 90° angles.

If the two 270° angles were separated by more than three 90° angles, the result would revert to one of the cases already identified above. Consequently, these are the only four possible cases. These four cases are shown in **Figure 6**.

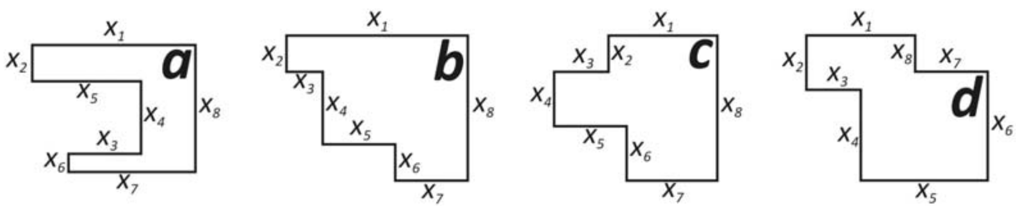

Figure 6. The four possible arrangements of rectangular lattice octagons.

Type (a) seems to be different from the others in the sense that it is U-shaped, suggesting some sort of concavity property. (The octagon with a hollow interior in **Figure 5** above has this same basic shape.) We can formalize this mathematically as follows. Define a set in the plane to be *HV-convex* provided that all vertical and all horizontal line segments which have both endpoints in the set do not leave the set. Thus, lattice octagons of type (a) are not HV-convex, while types (b), (c), and (d) are.

Suppose each octagon shown above is a lattice octagon with side lengths 1, 2, 3, 4, 5, 6, 7, and 8 units. Then the perimeter of every lattice octagon is 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36 units. Consequently, lattice octagons of type (b) are impossible, because it implies that $x_1 + x_8 = 18$, yet the maximum possible value of two side lengths is 7 + 8 = 15. Further, lattice octagons of type (a) can be obtained from type (c) by arranging segments x_3, x_4, and x_5 inside the octagon so that it is not HV-convex. (Note that this is only possible if $x_5 < x_1$ and $x_3 < x_7$, and these restrictions will be taken into account when octagons of type (a) are counted.) In view of these two realizations, it becomes clear that figures (c) and (d) are the only two arrangements that need to be considered, and octagons of type (a) can be treated as complementary octagons of type (c).

One of the important discoveries that students make when exploring the Lattice Octagon Problem is realizing that every lattice octagon can be enclosed by a rectangle. The smallest possible rectangle that can enclose a lattice octagon is called its *frame*. Lattice octagons of types (c) and (d) can be derived directly from the two types of frames (C) and (D) shown in **Figure 7**.

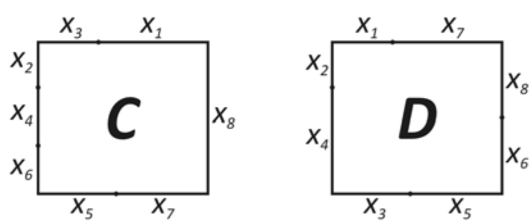

Figure 7. Possible frames (C) and (D) for lattice octagons of types (c) and (d).

Two lattice octagons are congruent if they can be superimposed after rotation or reflection. To avoid counting these congruency duplications, we make the following assumptions about the frames for types (c) and (d). For rectangles of type (c), we assume that the maximum value in the set $\{x_1, x_3, x_5, x_7\}$ is either x_1 or x_3. For rectangles of type (d), we assume $x_8 = 8$. With these restrictions, **Table 1** describes all possible arrangements of side lengths for rectangles of types (c) and (d).

Table 1.

All Possible Arrangements of Side Lengths for Octagons of Types (c) and (d)

Octagons Associated with Rectangles of Type (C)				Octagons Associated with Rectangles of Type (D)			
$\{x_2, x_4, x_6\}$	$\{x_1, x_3\}$	$\{x_5, x_7\}$	x_8	$\{x_2, x_4\}$	$\{\{x_1, x_7\}, \{x_3, x_5\}\}$	x_6	x_8
{1,2,3}	{4,8}	{5,7}	6	{2,7}	{{3,6},{4,5}}	1	8
{1,2,4}	{3,8}	{5,6}	7	{3,6}	{{2,7},{4,5}}	1	8
{1,2,5}	{3,7}	{4,6}	8	{4,5}	{{2,7},{3,6}}	1	8
				{4,6}	{{1,7},{3,5}}	2	8
				{4,7}	{{1,6},{2,5}}	3	8
				{6,7}	{{1,4},{2,3}}	5	8

The variables used in type (c), which represent numbers, can be arranged in $3! \cdot 2! \cdot 2! = 24$ distinct ways. This is because the numbers x_2, x_4, and x_6 can be arranged in $3! = 6$ ways, the numbers x_1 and x_3 can be arranged in $2! = 2$ ways, and the numbers x_5 and x_7 can be arranged in $2! = 2$ ways. Also, each of these 24 arrangements gives one lattice octagon of type (c). Because, as shown in the table, there are three different ways to group the sides in creating an octagon, there are $24 \cdot 3 = 72$ noncongruent lattice octagons of type (c).

In order to go from lattice octagons of type (c) to lattice octagons of type (a), some restrictions must be placed on the numbers x_1, x_3, x_5, and x_7. As mentioned previously, it must be the case that $x_3 < x_7$, which is equivalent to $x_5 < x_1$, since $x_1 + x_3 = x_5 + x_7$. Therefore, since the largest value of $\{x_1, x_3, x_5, x_7\}$ lies in the set $\{x_1, x_3\}$, we can construct a lattice octagon of type (a) from type (c) if and only if $x_1 > x_3$. This allows the variables in type (c) to be arranged in $3! \cdot 2! = 12$ different ways. Thus, there are $12 \cdot 3 = 36$ noncongruent lattice octagons of type (a).

Finally, observe that to go from rectangle (D) to lattice octagon (d), two opposite corners must be collapsed. The lattice octagon (d') in Figure 8 is given to emphasize that either pair of opposite corners of the rectangle (D) must be considered for collapsing. Of course, $x_8 = 8 > x_2$ and $x_8 > x_4$. Also, $x_6 < x_2$ and $x_6 < x_4$. This leads to four cases for type (d).

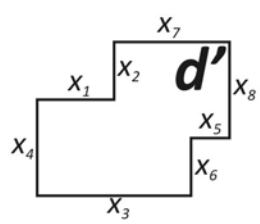

Figure 8. A lattice octagon of type (d').

Case 1. $x_1 = \max\{x_1, x_3, x_5, x_7\}$. In this case, we can collapse either pair of opposite corners in order to create a lattice octagon of type (d) or (d'). This yields $2 \cdot 2! \cdot 2! = 8$ lattice octagons.

- **Case 2.** $x_7 = \max\{x_1, x_3, x_5, x_7\}$. In this case, we can collapse only the upper left and lower right corners to create a lattice octagon of type (d'). This yields $1 \cdot 2! \cdot 2! = 4$ lattice octagons.
- **Case 3.** $x_3 = \max\{x_1, x_3, x_5, x_7\}$. In this case, we can collapse only the upper left and lower right corners to create a lattice octagon of type (d'). This yields $1 \cdot 2! \cdot 2! = 4$ lattice octagons.
- **Case 4.** $x_5 = \max\{x_1, x_3, x_5, x_7\}$. In this case, we can collapse either pair of opposite corners to create a lattice octagon of type (d) or (d'). This yields $2 \cdot 2! \cdot 2! = 8$ lattice octagons.

Combining cases 1 through 4, we see that rectangle (D) can be used to create a total of 8 + 4 + 4 + 8 = 24 non-congruent lattice octagons of type (d). Because, as shown in the table, there are six different ways to group the sides in creating an octagon, we have a total of $24 \cdot 6 = 144$ noncongruent lattice octagons of type (d).

Table 2.

Number of Lattice Octagons of Types (a)–(d).

Type	**Number of Lattice Octagons**
(a)	36
(b)	0

(c)	72
(d)	144
Total	252

Table 2 indicates that there are 252 noncongruent lattice octagons. This result answers Question 4. Returning to the list of five questions posed at the beginning of this section, we can now answer several others.

For Question 1, the lattice octagon shown on the electronic geoboard in **Figure 2** shows that the segments 1, 2, 3, …, 8 can be placed in order.

For Question 2, note that the largest rectangle with perimeter 36 is a 9 × 9 square whose area is 81 square units. Two rectangles can be removed from the corners, with dimensions 2 × 3 and 1 × 4. This forms a lattice octagon of type (d), and its area is 71 square units. This octagon is represented by the arrangement described in the first row of Table 1 for octagons associated with rectangles of type (D).

For Question 5, the answer is simply no. If segments with lengths 1 through 10 units were used to construct a decagon, the perimeter would be 1 + 2 + 3 + … + 10 = 55 units. As shown above, the sum of the side lengths on opposite sides of the frame must be equal, so the number of units in the perimeter must be even.

The solution to Question 3 is left for the reader.

REFERENCES

National Council of Teachers of Mathematics. 2009. *Daily Puzzle Challenge*.

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