The Space of Closed Subsets of a Convergent Sequence

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Many topological spaces are simply sets of points(atoms) endowed with a topology. Some spaces, however, have elements that are functions, matrices, or other non-atomic items. Another special type of space has elements that are themselves subsets of another space; these spaces are called *hyperspaces*. Hyperspaces are metric spaces, and the metric defined on them is called the *Hausdorff* metric. Hyperspaces whose points are the closed subsets and hyperspaces whose points are the closed *connected* subsets (of metric spaces) have been extensively studied. Also, hyperspaces of closed and convex subsets of a bounded convex set in euclidean space are of great interest in geometry. See Lay[3]. In recent years, geometers and topological dynamicists have explored spaces of closed and bounded subsets of the plane in connection with the study of fractals. One of the major results in the theory is that the hyperspace of closed subsets of a closed interval of real numbers is homeomorphic with the Hilbert cube, and with the space I^∞, the countable product of unit intervals. For more general information about the Hausdorff metric and spaces of subsets, see Devaney [2] and Sieradski [6].

The main result of this paper is a topological characterization of the space of closed subsets of a convergent sequence of points. The proof given here provides a *homeomorphic* embedding of the space in the plane, E_2. The result was first proved by Pelczynski[5]. He studied arbitrary compact zero-dimensional metric spaces, so his proofs are much more widely applicable, but they are also somewhat technical. Our proof depends only on some well known results in the theory of metric spaces, and is therefore accessible to advanced undergraduate mathematics majors. We also prove that for no convergent sequence of real numbers is there an *isometric* embedding of the hyperspace in euclidean space, E_n, for any n. For other results and discussion, see Nadler[4].

Let (X, d) denote a compact metric space. The *hyperspace* $(2^X, D)$ of X is the metric space whose points are the closed nonempty subsets of X and whose metric is the Hausdorff metric D given by

$$D(A, B) = inf\{\epsilon > 0 : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\},$$

where

$$N_\epsilon(A) = \{x \in X|\ d(a, x) < \epsilon \text{ for some } a \in A\}.$$

For example, let $A = \{(x, y)|\ x = 0 \text{ and } 0 \le y \le 20\}$ and $B = \{(x, y)|\ x = 10 \text{ and } 10 \le y \le 20\}$. Figure 1 shows that $\forall \epsilon > 0, N_{10+\epsilon}(A) \supset B$ and $N_{10\sqrt{2}+\epsilon}(B) \supset A$. However, $N_{10\sqrt{2}}(B) \not\supset A$, so $D(A, B) = 10\sqrt{2}$.

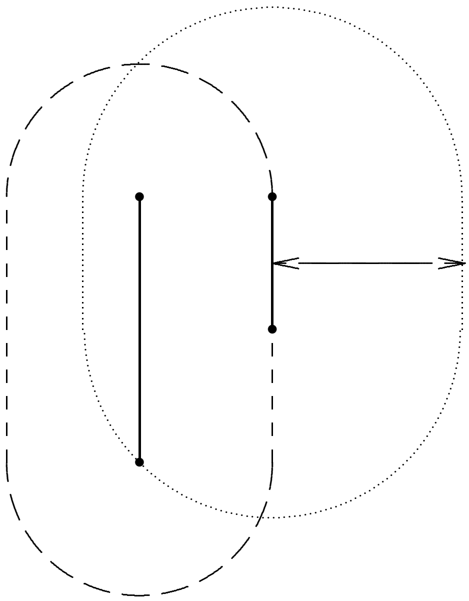

Figure 1. The distance function D, where A and B are vertical segments.

It is well known that 2^X is compact if and only if X is compact and that the topology of 2^X depends only on the topology of X and not on the metric for X. Also, since each point of X is a closed subset, X is a subset of 2^X, and the copy of X embedded in 2^X is isometric with X itself since the Hausdorff metric D for 2^X restricted to the subset of singletons is identical to the original metric d for X. See Nadler[4] for this result and more. Pelczynski investigated the hyperspace of $K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with the metric it inherits from the real line. A corollary to his theorem answers the question, "topologically, what is 2^K?"

Pelczynski's Theorem *Let X be a zero-dimensional infinite compact metric space with a dense set of isolated points. Then the space 2^X is homeomorphic with the subset $T(C)$ of the product space $[0,1] \times C$ given by*

$$T(C) = (0, C) \cup \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{N(k)} (k^{-1}, x_{k,n}),$$

where $\bigcup_{n=1}^{N(k)} \{x_{k,n}\}$ is an arbitrary fixed k^{-1}-net for C, and C is the Cantor discontinuum.[1]

Now we can state the corollary that is our first theorem.

THEOREM 1. The space 2^K is homeomorphic with a space $Y = Y_1 \cup Y_2 \subset E_2$ where Y_1 is homeomorphic with the Cantor ternary set and Y_2 is a countable set, every point of which is isolated in $Y = Y_1 \cup Y_2$ and such that every point of Y_1 is the limit of a sequence of points of Y_2.

[1]By a k^{-1}-net for C, Pelczynski means a set S of points in $T(C)$ such that every point of C is within k^{-1} of some point of S.

Pelczynski also showed that such a space is unique up to homeomorphism. Proof of Theorem 1. First, consider which subsets of K are closed. Clearly, every finite subset is closed. Which infinite ones are closed? Precisely those that contain 0. It is helpful to view 2^K as the union $Z \cup F$ where Z is the family of all subsets of K that contain 0, and F is the set of nonempty finite subsets that do not contain 0. Let C denote the Cantor ternary set. That is, $C = \{x \in \mathbf{R} : 0 \leq x \leq 1$ and x has a ternary representation consisting only of 0's and 2's$\}$. Next we embed 2^K in the plane. We define a function $\varphi : 2^K \to E_2$ in terms of a function f from Z to sequences of 0's and 2's: For each $U \in Z$

$$(f(U))_i = \begin{cases} 0 & if\ \frac{1}{i} \notin U \\ \\ 2 & if\ \frac{1}{i} \in U \end{cases}$$

Then

$$\varphi(U) = (\varphi_1(U), \varphi_2(U)) = \left(\sum_{i=1}^{\infty} f(U)_i \cdot 3^{-i}, 0 \right).$$

Each finite set not containing zero is mapped as follows. If $U \in F$, order the elements of U from smallest to largest: $a_1 < a_2 < \cdots < a_n$. Then define

$$\varphi(U) = (\varphi_1(U \cup \{0\}), a_1).$$

In other words, $\varphi(U \cup \{0\})$ is a point of C and $\varphi(U)$ is a point just above $\varphi(U \cup \{0\})$ in the plane at a height equal to the smallest element of U. For example

$$\varphi(\{0, 1, \frac{1}{2}, \frac{1}{3}\}) = (\frac{26}{27}, 0),$$

and

$$\varphi(\{1, \frac{1}{2}, \frac{1}{3}\}) = (\frac{26}{27}, \frac{1}{3}).$$

Thus the finite subsets of K are mapped to left endpoints of C if they have 0 as a member and to points above the x-axis otherwise. Our task is to prove that this mapping φ is a homeomorphism. Since 2^K is compact, it is enough to show that φ is both one-to-one and continuous. To see that φ is one-to-one, take distinct points U and V of 2^K. We will show that $\varphi(U) \neq \varphi(V)$. If one belongs to Z and the other to F, $\varphi_2(U) \neq \varphi_2(V)$. If both belong to F, there exists k such that $\frac{1}{k}$ belongs to $(U \setminus V) \cup (V \setminus U)$. If k_0 is the smallest such k and $\frac{1}{k_0} \in U$, then $\varphi_1(U) > \varphi_1(V)$. Similarly, if both U and V belong to Z, then $\varphi_1(U) \neq \varphi_1(V)$.

To see that φ is continuous, let $\{U_n\}_{n=1}^{\infty}$ be a convergent sequence of elements of 2^K and suppose $\lim_{n\to\infty} U_n = U_0$. To see that

$$\lim_{n\to\infty} \varphi(U_n) = \varphi(U_0),$$

we distinguish two cases:
Case 1: $0 \notin U_0$. In this case, $\{U_n\}$ is eventually constant.
Case 2: $0 \in U_0$. We show A, $\varphi_1(U_n) \to \varphi_1(U_0)$ and B, $\varphi_2(U_n) \to \varphi_2(U_0)$.

A. For each $k \in \mathbf{N}$, let $I_k = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}\}$. We prove that for each $k \in \mathbf{N}$ there exists $m_0 \in \mathbf{N}$ such that if $m > m_0$ then $I_k \cap U_0 = I_k \cap U_m$. Suppose not. Then there exists $j \leq k$ such that $\frac{1}{j} \in U_0$ and $\frac{1}{j} \notin U_m$ (or vice-versa), in which case

$$D(U_0, U_m) \geq \left|\frac{1}{j} - \frac{1}{j+1}\right| = \left|\frac{j+1-j}{j(j+1)}\right| = \frac{1}{j(j+1)} \geq \frac{1}{(k+1)^2}.$$

This contradicts the convergence of the sequence $\{U_n\}$. Now we prove that $\varphi_1(U_n) \to \varphi_1(U_0)$. Let $\epsilon > 0$ be given. There exists a positive integer m_0 such that $\sum_{i=m_0}^{\infty} 2 \cdot 3^{-i} < \epsilon$. Now pick m_0 large enough so that if $m > m_0$ then $I_{m_0} \cap U_0 = I_{m_0} \cap U_m$. Let $D = \{i | \frac{1}{i} \in U_m \setminus U_0 \cup U_0 \setminus U_m\}$. Then $|\varphi_1(U_m) - \varphi_1(U_0)| = \sum_{k \in D} 2 \cdot 3^{-k} \leq \sum_{k=m_0}^{\infty} 2 \cdot 3^k < \epsilon$.

B. To see that $\varphi_2(U_n) \to \varphi_2(U_0)$, let $\epsilon > 0$ be given. We can take m_0 large enough that for any $m > m_0$, U_m contains a point $\frac{1}{k}$ less than ϵ. Then $\varphi_2(U_m) \leq \frac{1}{k} < \epsilon$. This completes the proof.

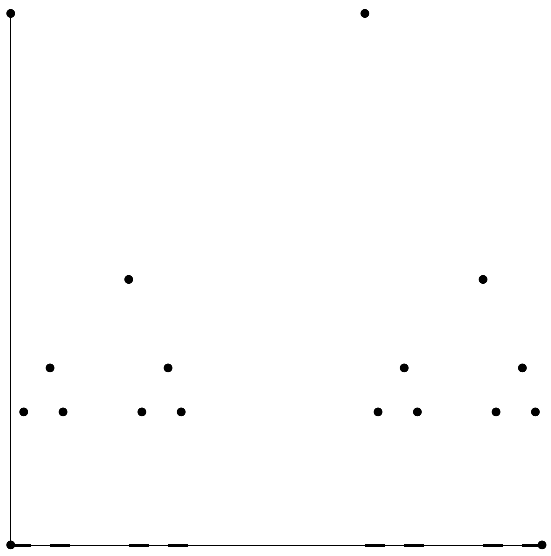

Figure 2. Part of the image of φ.

Rather than defining a homeomorphism into the plane, we could use a modified function to map 2^K into the line by mapping each set not containing 0 to the middle of the interval above whose right endpoint it lies. For example, $\varphi(\{1\}) = \frac{1}{2}$.

Our second theorem shows that it is not possible to improve on the homeomorphic embedding of 2^K in the plane given in the proof of theorem 1.

THEOREM 2. Let $\{a_i\}_{i=1}^{\infty}$ be a decreasing sequence of real numbers with $a_1 \leq 1$ and $\lim_{n\to\infty} a_i = 0$, and let $I = \{0, a_1, a_2, \ldots\}$. Then, the hyperspace $(2^I, D)$ is not isometrically embeddable in euclidean space E_n for any n.

Proof. Suppose there is such an embedding. Let $b = a_1$ and let a be a point of the sequence satisfying $a < \frac{b}{2}$. Consider the six elements $\{0\}, \{0, b\}, \{b\}, \{a\}, \{a, b\}$, and $\{0, a\}$ of 2^K and denote their images in E_n under the embedding as $\{0\}', \{0, b\}', \{b\}'$, $\{a\}', \{a, b\}'$, and $\{0, a\}'$. See Figure 3. If A, B and C are three distinct points of the plane for which $D(A, C) = D(A, B) + D(B, C)$, then B lies on the segment from A to C. Using this fact, it follows that $\{a\}'$ belongs to the segment determined by $\{0\}'$ and $\{b\}'$, $\{a, b\}'$ belongs to the segment determined by $\{b\}'$ and $\{0, b\}'$, and $\{0, a\}'$ belongs to the segment determined by $\{0\}'$ and $\{0, b\}'$. Now $\{0\}', \{0, b\}'$, and $\{b\}'$ are the vertices of an equilateral triangle, as are $\{b\}', \{a\}', \{a, b\}'$ and $\{0\}', \{a\}'$, and $\{0, a\}'$. Since $D(\{a, b\}, \{0, a\}) = b - a$ and $D(\{0, b\}, \{0, a\}) = b - a$, we can see that $\angle\{0, b\}'\{a, b\}'\{0, a\}' = \angle\{a, b\}'\{0, b\}'\{0, a\}' = 60°$. The triangle $\{a, b\}'\{0, b\}'\{0, a\}'$ is equilateral, so $D(\{a, b\}, \{0, a\}) = D(\{a, b\}, \{0, b\}) = b - a$, but $D(\{a, b\}, \{0, b\}) = a$ which contradicts our assumption that $a < \frac{b}{2}$.

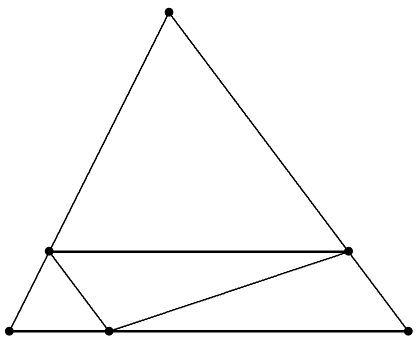

Figure 3. The triangle with vertices $\{0\}', \{b\}'$ and $\{0, b\}'$ in the hyperspace 2^K.

Challenge 1: Find a formula for the distance between two subsets A and B based on the size of the first integer i for which $\frac{1}{i}$ belongs to one of the sets but not the other.

Challenge 2: Show that theorem 1 follows no matter what metric is given for the

convergent sequence.
Challenge 3: Show that theorem 2 follows no matter what metric is given for the convergent sequence.
Challenge 4: If we use the alternative embedding of 2^K in the line, the homeomorphism establishes a linear ordering on 2^K. Describe this ordering explicitly.
Challenge 5: What goes wrong if we define φ using powers of 2 instead of powers of 3 as follows:

$$(f(U))_i = \begin{cases} 0 & if\ \frac{1}{i} \notin U \\ \\ 1 & if\ \frac{1}{i} \in U \end{cases} \quad \text{and } \varphi(U) = \left(\sum_{i=1}^{\infty} f(U)_i \cdot 2^{-i}, 0 \right).$$

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References

[1] M. F. Barnsley, *Fractals Everywhere*, Academic Press, New York, 1988.

[2] R. L. Devaney, *An Introduction to Chaotic Dymanical Systems*, Benjamin Cummings, Menlo Park, CA., 1986.

[3] S. R. Lay, *Convex Sets and their Applications*, Wiley Interscience, New York, 1982.

[4] Sam B. Nadler, *Hyperspaces of Sets*, Marcel Dekker, Inc., New York, 1978.

[5] A. Pelczynski, A Remark of Spaces 2^X for Zero-dimensional X, *Bulletin of the Polish Academy of Sciences*, 13, (1965) 85-89.

[6] A. J. Sieradski, *An Introduction to Topology and Homotopy*, PWS-Kent, Boston, 1992.