The poem Going To St Ives sparked this short essay on how arithmetic can lead to an understanding of a beautiful algebraic idea. A great third grade teacher can do the arithmetic in such a way that his students recognize four or five years later the beautiful ideas associated with summing a finite or infinite geometric series. Today we discuss one reason elementary teachers should have a firm grasp of some ideas that students normally encounter in algebra. This essay was inspired by a message from Roger Howe.

> As I was going to St. Ives,
> I met a man with 7 wives.
> Each wife had 7 sacks.
> Each sack had 7 cats.
> Each cat had 7 kits.
> Kits, cats, sacks and wives,
> How many were going to St. Ives?

The earliest known published version of this riddle comes from a manuscript dated to around 1730 (but it differs in referring to 'nine' rather than 'seven' wives). The modern form was first printed around 1825. There are a number of places called St Ives in England and elsewhere. It is generally thought that the rhyme refers to St Ives, Cornwall, when it was a busy fishing port and had many cats to stop the rats and mice destroying the fishing gear, although some people argue it was St Ives, Huntingdonshire as this is an ancient market town and therefore an equally plausible destination. The trick answer is 1 since the argument could be made that only the poser was 'on his way' to St Ives. But the intention here is to ask the sum

$$7 + 7^2 + 7^3 + 7^4.$$

For the moment we leave out the man himself. Of course we can solve it as a plain arithmetic problem with big numbers, and it is tempting to simply crunch the numbers on a calculator. But suppose the third grade teacher has a firm grasp of geometric series. He might write, let S denote the sum we seek. Then

$$S = 7 + 7^2 + 7^3 + 7^4.$$

Now compute $7S$. Of course we get $7S = 7(7 + 7^2 + 7^3 + 7^4) = 7^2 + 7^3 + 7^4 + 7^5$.

Next consider $7S - S$.

$$\begin{aligned} 7S - S &= 7^2 + 7^3 + 7^4 + 7^5 - (7 + 7^2 + 7^3 + 7^4) \\ &= 7^5 + 7^4 - 7^4 + 7^3 - 7^3 + 7^2 - 7^2 - 7 \\ &= 7^5 - 7 = 7(7^4 - 1) \\ &= 7(7^2 - 1)(7^2 + 1) \\ &= 7 \cdot 48 \cdot 50 = 6 \cdot 2800 \end{aligned}$$

So $S = 2800$. Adding in the man himself, we get 2801 things going to St Ives.

Where is the algebra? So far, there isn't any.

But let's replace the integer 7 with the letter x, and see how what happens.

$$S = x + x^2 + x^3 + x^4.$$

Now compute xS. Of course we get $xS = x(x + x^2 + x^3 + x^4) = x^2 + x^3 + x^4 + x^5$. Next consider $xS - S$.

$$\begin{aligned} xS - S &= x^2 + x^3 + x^4 + x^5 - (x + x^2 + x^3 + x^4) \\ &= x^5 + x^4 - x^4 + x^3 - x^3 + x^2 - x^2 - x \\ &= x^5 - x = x(x^4 - 1) \\ &= x(x^2 - 1)(x^2 + 1) \end{aligned}$$

This leads to the surprisingly lovely formula

$$S = \frac{x(x-1)(x+1)(x^2+1)}{x-1} = x(x+1)(x^2+1).$$

Now we have a solution to any of the possible variations of the riddle. In case $x = 7$, of course we get 2800, but we can replace x with any value, say 9 as in the original riddle. The point is that if the elementary teacher is aware that eventually his students are going to need to understand the second general solution, he can pave the way by discussing the first solution above.

Let us see if we can milk this idea. A finite geometric series is a sum of the form

$$S = a + ar + ar^2 + \cdots + ar^n.$$

Each term after the first is r times the previous term. Let's try again the idea that worked above.

$$\begin{aligned} rS - S &= r(a + ar + ar^2 + \cdots ar^n) - (a + ar + ar^2 + \cdots ar^n) \\ &= ar + ar^2 + ar^3 + \cdot + ar^{n+1} - (a + ar + ar^2 + \cdots ar^n) \\ &= ar^{n+1} - a \end{aligned}$$

Thus we have

$$S = \frac{ar^{n+1} - a}{r - 1}.$$

In the case where $|r| < 1$ the infinite series converges an we have

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \lim_{n\to\infty} \frac{ar^{n+1} - a}{r - 1} = \frac{a}{1 - r}.$$

The reason is that as $n \to \infty$, $r^n \to 0$.

But we are not yet done. Compute the infinite sum $S = 1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots$, where the n^{th} term is $\frac{n}{2^n}$. Here we can multiply S by $1/2$ to get $S = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$, which we can then subtract from S to get $S - S/2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which we can quickly see is just 2. Hence, $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots = 4$.