Invariant Relations for the Derivatives of Two Arbitrary Polynomials

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Let $P(x)$ and $Q(x)$ be polynomials of degrees n and m respectively. Let $P^i(x)$ and $Q^j(x)$ denote the ith, jth derivatives of $P(x)$ and $Q(x)$. Define the $(m+n)\times(m+n)$ matrix $M(x)$ as follows.

1. Each row $i, 1 \le i \le m$, of $M(x)$ is the following.

$$\overbrace{0,0,\cdots,0}^{\leftarrow i-1 \rightarrow}, \frac{P^n(x)}{n!}, \frac{P^{n-1}(x)}{(n-1)!}, \cdots, \frac{P'(x)}{1!}, P(x), \overbrace{0,0,\cdots,0}^{\leftarrow m-i \rightarrow}.$$

2. Each row $m+i, 1 \le i \le n$, of $M(x)$ is the following.

$$\overbrace{0,0,\cdots,0}^{\leftarrow i-1 \rightarrow}, \frac{Q^m(x)}{m!}, \frac{Q^{m-1}(x)}{(m-1)!}, \cdots, \frac{Q'(x)}{1!}, Q(x), \overbrace{0,0,\cdots,0}^{\leftarrow n-i \rightarrow}.$$

We show that the determinant $|M(x)|$ of $M(x)$ has a value that is independent of x. Thus, we can call $|M(x)|$ an invariant. We will give the reader all of the necessary background material and this will make the paper accessible to almost any undergraduate mathematics student. Also at the end we give some specific examples.

1 The Resultant of Two Polynomials

The resultant $\rho(P(x), Q(x))$ of two polynomials $P(x), Q(x)$ is the standard determinant given in Axiom 1 which gives by its zero or non-zero value the necessary and sufficient condition so that $P(x)$ and $Q(x)$ have no roots in common.

Also, if $P(x) = A_n \cdot \prod_{i=1}^{n}(x - r_i)$ and $Q(x) = B_m \cdot \prod_{i=1}^{m}(x - s_i)$, then $\rho(P(x), Q(x)) = A_n^m B_m^n \prod (r_i - s_j)$. If this last property is taken as a definition, then the following Axiom 1 is a standard property of resultants that is proved in the theory of equations. See pp. 99-104, [2] for the details. Of course, the reader will immediately see the similarity of Axiom 1 and the determinant $|M(x)|$ that was given in the Introduction.

In Axiom 1 and Theorem 1, we use the notation $P(x) = \sum_{i=0}^{n} A_i x^i = A_n \cdot \prod_{i=1}^{n} (x - r_i), A_n \neq 0$, and $Q(x) = \sum_{i=0}^{m} B_i x^i = B_m \cdot \prod_{i=1}^{m} (x - s_i), B_m \neq 0$.

Axiom 1 *$\rho(P(x), Q(x))$ equals the determinant of the $(m+n) \times (m+n)$ matrix M defined as follows.*

1. *Each row $i, 1 \leq i \leq m$, of M is defined as follows.*

$$\overbrace{0, 0, \cdots, 0}^{\leftarrow i-1 \rightarrow}, A_n, A_{n-1}, \cdots, A_1, A_0, \overbrace{0, 0, \cdots, 0}^{\leftarrow m-i \rightarrow}.$$

2. *Each row $m+i, 1 \leq 1 \leq n$, of M is defined as follows.*

$$\overbrace{0, 0, \cdots, 0}^{\leftarrow i-1 \rightarrow}, B_m, B_{m-1}, \cdots, B_1, B_0, \overbrace{0, 0, \cdots, 0}^{\leftarrow n-i \rightarrow}.$$

Axiom 1 is a theorem in the elementary theory of equations. See [2]. We now illustrate Axiom 1. First, suppose $P(x) = (x-a)(x-b) = x^2 - (a+b)x + ab$ and $Q(x) = x - c$. Then $\rho(P(x), Q(x)) = (a-c)(b-c)$. Also, by Axiom 1, $\rho(P(x), Q(x)) =$

$$\begin{vmatrix} 1 & -(a+b) & ab \\ 1 & -c & 0 \\ 0 & 1 & -c \end{vmatrix}$$

$= c^2 - (a+b)c + ab = (a-c)(b-c)$.

Second, suppose $P(x) = (x-a)(x-b)$ and $Q(x) = (x-c)(x-d)$. Then $\rho(P(x), Q(x)) = (a-c)(b-c)(a-d)(b-d)$. Also, by Axiom 1, $\rho(P(x), Q(x)) =$

$$\begin{vmatrix} 1 & -(a+b) & ab & 0 \\ 0 & 1 & -(a+b) & ab \\ 1 & -(c+d) & cd & 0 \\ 0 & 1 & -(c+d) & cd \end{vmatrix}$$

Theorem 1 *For all complex numbers b,*

$$\rho(P(x+b), Q(x+b)) = \rho(P(x), Q(x)).$$

Proof. Of course, $r_1, r_2, \cdots, r_n$ are the roots of $P(x)$ and $s_1, s_2, \cdots, s_m$ are the roots of $Q(x)$. Also, let $\overline{r}_1, \overline{r}_2, \cdots, \overline{r}_n$ be the roots of $P(x+b)$ and let $\overline{s}_1, \overline{s}_2, \cdots, \overline{s}_m$ be the roots of $Q(x+b)$. Now each $\overline{r}_i = r_i - b$ and each $\overline{s}_j = s_j - b$.

Also, $\rho(P, Q) = A_n^m B_m^n \prod (r_i - s_j)$. Therefore, $\rho(P(x+b), Q(x+b)) =$
$A_n^m B_m^n \prod (\overline{r}_i - \overline{s}_j) = A_n^m B_m^n \prod (r_i - s_j) = \rho(P, Q)$. ■

2 Proving the Theorem Given in the Introduction

As always, let $P(x) = \sum_{i=0}^{n} A_i x^i,\ A_n \neq 0$. Now $P(x) = \sum_{i=0}^{n} \frac{P^i(b)}{i!}(x-b)^i$. Therefore,
$(*)\ P(x+b) = \sum_{i=0}^{n} \frac{P^i(b)}{i!} x^i$.

Likewise, if $Q(x) = \sum_{i=0}^{m} B_i x^i, B_m \neq 0$, then $(**)\ Q(x+b) = \sum_{i=0}^{m} \frac{Q^i(b)}{i!} x^i$.

From Theorem 1, $\rho(P(x+b), Q(x+b)) = \rho(P(x), Q(x))$ which means that $\rho(P(x+b), Q(x+b))$ has a value that is independent of b.

If we now use Axiom 1 with $(*), (**)$ to evaluate $\rho(P(x+b), Q(x+b))$, we immediately see that the theorem given in the Introduction is true where we are now using the variable 'b' instead of the variable 'x'.

3 Some Specific Examples

Suppose $P(x) = \sum_{i=0}^{2} A_i x^i = A_2x^2 + A_1x + A_0, Q(x) = \sum_{i=0}^{2} B_i x^i = B_2x^2 + B_1x + B_0$.

Of course, $P(x)$ and $Q(x)$ are second degree polynomials and $P'(x), Q'(x)$ are first degree polynomials. Therefore, by using the theorem with each pair $(P, Q), (P', Q), (P, Q'), (P', Q')$, we have the following four invariants involving the derivatives of $P(x), Q(x)$.

In (2), (3), (4), we note that $(P')' = P'', (Q')' = Q''$.

1. $\begin{vmatrix} \frac{P''(x)}{2} & P'(x) & P(x) & 0 \\ 0 & \frac{P''(x)}{2} & P'(x) & P(x) \\ \frac{Q''(x)}{2} & Q'(x) & Q(x) & 0 \\ 0 & \frac{Q''(x)}{2} & Q'(x) & Q(x) \end{vmatrix}$,

2. $\begin{vmatrix} P''(x) & P'(x) & 0 \\ 0 & P''(x) & P'(x) \\ \frac{Q''(x)}{2} & Q'(x) & Q(x) \end{vmatrix}$,

3. $\begin{vmatrix} \frac{P''(x)}{2} & P'(x) & P(x) \\ Q''(x) & Q'(x) & 0 \\ 0 & Q''(x) & Q'(x) \end{vmatrix}$,

4. $\begin{vmatrix} P''(x) & P'(x) \\ Q''(x) & Q'(x) \end{vmatrix}$.

Note that we can also include anti-derivatives e.g., $\int P,\ \int\int P,\ \int Q$, etc. If we make $P(x)$ simple and let $Q(x)$ be arbitrary, then we can write down invariants that can easily be evaluated.

For example, if $P(x) = x - b$ and $Q(x)$ is a cubic, then we have the following invariant:

$$\begin{vmatrix} 1 & x-b & 0 & 0 \\ 0 & 1 & x-b & 0 \\ 0 & 0 & 1 & x-b \\ \frac{Q'''(x)}{6} & \frac{Q''(x)}{2} & Q'(x) & Q(x) \end{vmatrix}$$

$$= Q(x) + Q'(x)(b-x) + \frac{Q''(x)}{2}(b-x)^2 + \frac{Q'''(x)}{3!}(b-x)^3 = Q(b).$$

This is the standard Taylor's series if we interchange x and b. Finally, suppose $\overline{P}(x)$ is an arbitrary polynomial of degree n. Define $P(x) = \overline{P}^i(x), Q(x) = \overline{P}^j(x)$. Then by varying $i, j \in \{0, 1, 2, \cdots, n-1\}, i < j$, we can create C_2^n different invariants that involve the derivatives of $\overline{P}(x)$.

4 A Research Problem

Suppose

$$P(x) = \sum_{i=0}^{n} A_i x^i, A_n \neq 0,$$

and

$$\overline{P}(x) = \sum_{i=0}^{n} B_i x^i, B_n \neq 0,$$

are two nth degree polynomials. We say that P and $\overline{P}$ are *weakly congruent* (denoted $P(x) \cong \overline{P}(x)$) if $\overline{P}(x) = P(x+b)$ for some complex number b. It is easy to show that $\cong$ is reflexive, symmetric and transitive, and therefore an equivalence relation for the collection of all degree n polynomials. We can see quickly that $A_n = B_n$. We wish to find a collection of $n-1$ polynomials, $P_3(x_1, x_2, x_3), P_4(x_1, x_2, x_3, x_4), P_5(x_1, x_2, x_3, x_4, x_5), \ldots, P_{n+1}(x_1, x_2, \ldots, x_{n+1})$ such that $P(x) \cong \overline{P}(x)$ if and only if

1. $A_n = B_n$
2. $P_3(A_n, A_{n-1}, A_{n-2}) = P_3(B_n, B_{n-1}, B_{n-2})$
3. $P_4(A_n, A_{n-1}, A_{n-2}, A_{n-3}) = P_4(B_n, B_{n-1}, B_{n-2}, B_{n-3})$

   $\vdots$

n. $P_{n+1}(A_n.A_{n-1}, \ldots, A_0) = P_{n+1}(B_n.B_{n-1}, \ldots, B_0)$.

We can call $P_3, P_4, \ldots, P_{n+1}$ invariants under $\cong$ that classify the equivalence relation. We now start the reader off.

If $P(x) \cong \overline{P}(x)$, then

$$\overline{P}(x) = \sum_{i=0}^{n} B_i x^i = P(x+b) = \sum_{i=0}^{n} \frac{P^i(b)}{i!} x^i.$$

Therefore, $B_i = \frac{P^i(b)}{i!}$. Also, $A_i = \frac{P^i(0)}{i!}$. It follows that $P^i(b) = i!B_i, P^i(0) = i!A_i$. For each $i, j \in \{0, 1, 2, \ldots, n-1\}, i < j$, if we call $x = b$, then we know that we can use each pair (P^i, P^j) to create an invariant (under $\cong$) that involve the coefficients $A_0, A_1, \ldots, A_n$, $B_0, B_1, \ldots, B_n$. These invariants are necessary conditions for $P(x) \cong \overline{P}(x)$. However, when $n \geq 3$, this collection is too large, and it overshoots the number of invariants required in the problem. So the problem is to cull out (with proof) a subcollection that solves the problem. As an example, let $P(x) = A_2x^2 + A_1x + A_0$, $\overline{P}(x) = B_2x^2 + B_1x + B_0$. Then $B_i = \frac{P^i(b)}{i!}, A_i = \frac{P^i(0)}{i!}, i = 0, 1, 2$. Using (P, P'), we have the invariant

$$\begin{vmatrix} \frac{P''(b)}{2} & P'(b) & P(b) \\ P''(b) & P'(b) & 0 \\ 0 & P''(b) & P'(b) \end{vmatrix} = \begin{vmatrix} \frac{P''(0)}{2} & P'(0) & P(0) \\ P''(0) & P'(0) & 0 \\ 0 & P''(0) & P'(0) \end{vmatrix}.$$

This gives

$$\begin{vmatrix} B_2 & B_1 & B_0 \\ 2B_2 & B_1 & 0 \\ 0 & 2B_2 & B_1 \end{vmatrix} = \begin{vmatrix} A_2 & A_1 & A_0 \\ 2A_2 & A_1 & 0 \\ 0 & 2A_2 & A_1 \end{vmatrix}$$

which gives $4B_0B_2^2 - B_1^2B_2 = 4A_0A_2^2 - A_1^2A_2$. Since $A_2 = B_2 \neq 0$, it follows that $4B_0B_2 - B_1^2 = 4A_0A_2 - A_1^2$. Equating $\overline{P}(x) = B_2x^2 + B_1x + B_0 = P(x+b) = A_2x^2 + (2A_2b + A_1)x + A_2b^2 + A_1b + A_0$, we see that $A_2 = B_2, b = \frac{B_1 - A_1}{2A_2}$. Also, we require $A_2\left[\frac{B_1-A_1}{2A_2}\right]^2 + A_1\left[\frac{B_1-A_1}{2A_2}\right] + A_0 = B_0$. Since $A_2 \neq 0$, this is true if and only if $(B_1^2 - 2B_1A_1 + A_1^2) + 2A_1(B_1 - A_1) + 4A_0A_2 = 4A_2B_0$. Since $A_2 = B_2$, this is true if and only if $B_1^2 - 4B_0B_2 = A_1^2 - 4A_0A_2$, and these are the same two conditions as are given above.

As the reader explores higher degree polynomials, he will begin to appreciate the theory in this paper. Ed Barbeau showed us how to solve this research problem in a different way. Thus the reader might like to try solving this in a different way.

Let us now define another relation. We say $P(x), \overline{P}(x)$ are *strongly congruent* (denoted by $P(x) \bowtie \overline{P}(x)$) if $P(x) = P(x+b) + a$ for some complex numbers a, b. We invite the reader to very slightly modify his solution to the weak congruence problem above to define and solve an analogous problem for strong congruence. Finally, the reader might like to discuss the relationship between strong congruence and geometric congruence.

5 Discussion

Since in general the two polynomials $\overline{P}(x)$ of degree n and $\overline{Q}(x)$ of degree m need not be correlated in any way, it seems to be a remarkable fact that we can write down so many different invariants involving the derivatives of $\overline{P}(x), \overline{Q}(x)$. By defining $P(x) = \overline{P}^i(x), i \in \{0, 1, 2, \cdots, n-1\}$, and $Q(x) = \overline{Q}^j(x), j \in \{0, 1, 2, \cdots, m-1\}$, and then using each pair $\left(\overline{P}^i(x), \overline{Q}^j(x)\right)$ we can write down $n \cdot m$ different invariants involving the derivatives of $\overline{P}(x), \overline{Q}(x)$. Thus, if $n = m = 100$, we would have ten thousand different invariants involving two unrelated polynomials $\overline{P}(x), \overline{Q}(x)$. This seems almost unbelievable. If we include anti-derivatives, we will have an infinite number.

References

[1] Barbeau, E.J., Polynomials, (Problem Books in Mathematics) (Paperback), Springer Verlag, New York, 1989.

[2] Weisner, Louis, Introduction to the Theory of Equations, The MacMillan Company, New York, 1949.