2006 MATH Challenge

For full credit you must **justify** your answers.

1. Let $f(x) = (x+1)^2 e^{2x}$. The fiftieth derivative $f^{(50)}(0)$ can be expressed in the form $k2^n$ where k is an odd integer and n is a positive integer. Find k and n.

   Solution: Recall the Maclaurin series for e^x is $1 + x + x^2/2! + x^3/3! + \cdots$. Therefore $e^{2x} = 1 + 2x + (2x)^2/2 + (2x)^3/3! \cdots$. This means that

   $$f(x) = (x^2 + 2x + 1)e^{2x} = (x^2 + 2x + 1)\sum_{i=0}^{\infty} \frac{(2x)^i}{i!}.$$

   On one hand, the coefficient of the 50$^{\text{th}}$ term of the series is $f^{(50)}(0)/50!$ and on the other hand, the x^{50} term is $x^2\frac{(2x)^{48}}{48!} + 2x\frac{(2x)^{49}}{49!} + \frac{(2x)^{50}}{50!}$ so the coefficient is $\frac{(2)^{48}}{48!} + 2\frac{(2)^{49}}{49!} + \frac{(2)^{50}}{50!}$. Now it follows that

   $$\begin{aligned} f^{(50)}(0) &= 50!\left(\frac{(2)^{48}}{48!} + 2\frac{(2)^{49}}{49!} + \frac{(2)^{50}}{50!}\right) \\ &= 50 \cdot 49 \cdot 2^{48} + 50 \cdot 2^{50} + 2^{50} \\ &= 25 \cdot 49 \cdot 2^{49} + 102 \cdot 2^{49} \\ &= 1327 \cdot 2^{49} \end{aligned}$$

2. The three points $(4, 14, 8, 14)$, $(6, 6, 10, 8)$ and $(2, 4, 6, 8)$ are vertices of a 4-dimensional cube in 4-space. Find the center of the cube.

   Solution: Let $A = (4, 14, 8, 14)$, $B = (6, 6, 10, 8)$ and $C = (2, 4, 6, 8)$, and let d denote the (Euclidean) distance function. Then $d(A, B) = \sqrt{108}$, $d(A, C) = \sqrt{144}$, and $d(B, C) = \sqrt{36}$, so ABC is a right triangle with hypotenuse AC, by the converse of the Pythagorean Theorem. Since the distances are in the ratio $1 : \sqrt{3} : 2$ the segment AC connects vertices at opposite corners of the cube. Therefore, the center of the cube is the midpoint of this segment, $(3, 9, 7, 11)$.

3. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-9}$.

   Solution: First note that $\frac{1}{4n^2-9} = \frac{1}{6}(\frac{1}{2n-3} - \frac{1}{2n+3})$. Note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-3} = -1 + 1 + 1/3 + 1/5 + 1/7 + \cdots$ while $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3} = 1/5 + 1/7 + \cdots$ so their difference is $1/3$. Thus, the sum of the series is $1/6 \cdot 1/3 = 1/18$.

4. Let $n \geq 1$ be fixed. Suppose n points are placed at random on a circle. Let $P(n)$ denote the probability that all n points lie on the same side of some diameter. Find $P(n)$. In particular, find $P(2)$ and $P(3)$.

   Solution: Let us first inscribe a regular $2m+1$-gon in this circle where $m \geq n-1$ and m is fixed. We will calculate $(*)$ the probability if we choose n of these $2m+1$ vertices at random then these n points will lie on the same side of some diameter.

   Now the number of different ways that n vertices can be chosen from these $2m+1$ vertices so that these n vertices will lie on the same side of some diameter equals $(2m+1)\binom{n-1}{m}$. To see this, orient the circle in the counter clock wise direction as shown. Therefore, if n points lie on the same side of a diameter of the oriented circle, then we can single out a first member of these n points and call it $\overline{0}$. Thus, the $\overline{0}$ can be chosen in $2m+1$ different ways, and once $\overline{0}$ is chosen the other $n-1$ points can be chosen in $\binom{n-1}{m}$ different ways. The probability required in $(*)$ equals

$$\frac{(2m+1)\binom{n-1}{m}}{\binom{n}{2m+1}} = \frac{n \cdot [m(m-1)(m-2)\cdots(m-(n-2))]}{2m(2m-1)(2m-2)\cdots(2m-(n-2))}.$$

   The solution to problem (a) is

$$\lim_{m\to\infty} \frac{n \cdot [m(m-1)(m-2)\cdots(m-(n-2))]}{2m(2m-1)(2m-2)\cdots(2m-(n-2))} = \frac{n}{2^{n-1}}.$$

   Thus, $P(2) = 1$ and $P(3) = 3/4$.

5. Let $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ denote the set of nonzero decimal digits. Note that D has $\binom{9}{4} = 126$ four-element subsets. How many of these subsets $\{a, b, c, d\}$ can be used to build a three-digit base d number $N = abc_d$ such that the difference between N and the number $\overline{N}$ obtained by reversing the digits of N is a multiple of 21? For example $123_8 - 321_8 = (1 \cdot 8^2 + 2 \cdot 8 + 3) - (3 \cdot 8^2 + 2 \cdot 8 + 1) = -128 + 2 = -126 = -6(21)$, so the set $\{1, 2, 3, 8\}$ is one of the sets we need to count. Of course the number d used for the base must be the largest of the four numbers.

   Solution: We use the notation abc_d to mean the number $a \cdot d^2 + b \cdot d + c$. If $N = abc_d$, then we count the set $\{a, b, c, d\}$ if

$$\begin{aligned} N - \overline{N} &= a \cdot d^2 + b \cdot d + c - (c \cdot d^2 + b \cdot d + a) \\ &= (a - c)d^2 + c - a \\ &= (a - c)(d - 1)(d + 1) = 21k \end{aligned}$$

   In order that $(a-c)(d-1)(d+1)$ is a multiple of 21, at least one of $a-c, d-1$, and $d+1$ must be 7. If $d = 8$, then $d - 1 = 7$ and $d + 1 = 9$, so any a, b, c works as long as they are all less than 8. There are $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ such sets. If $d + 1 = 7$ then $d - 1 = 5$ and we must have $a - c = \pm 3$ (since $d = 6$, we cannot have $a - c = 6$). So $\{a, c\} = \{1, 4\}$ or $\{a, c\} = \{2, 5\}$. There are 3 sets of the form $\{1, 4, 6, b\}$, and 3 more of the form $\{2, 5, 6, b\}$, with $b < 6$. Now if $a - c = \pm 7$, then $\{a, c\} = \{1, 8\}$ and $d = 9$. In this case, $(d - 1)(d + 1) = 80$, and $(a - c)(d - 1)(d + 1)$ is not a multiple of 21. Hence the number of subsets satisfying the requirements is $35 + 6 = 41$ (the two sets enumerated are disjoint).

6. Suppose some faces of a large wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. Is it possible that the number of unit cubes with some red paint is exactly $M = 2006$ larger than the number of cubes with some black paint? Find the smallest number $M \geq 2006$ for which there is such a cube and find a way to paint the faces so that the number of unit cubes with some red paint is exactly M larger than the number of cubes with some black paint.

   Solution: The answer is $M = 2046$. We paint black an opposite pair of faces of a $33 \times 33 \times 33$ cube. Note that the number r of red faces must be four, five, or six. There are two cases with four red faces: the black faces are adjacent or not. The four equations we get are $r = 6: \; M = n^2 - 12n + 8$; $r = 5: \; M = 4(n-1)^2$; $r = 4: \; M = 2(n-1)^2$ in case the black faces are adjacent and $M = 2n(n-2)$ in case the black faces are opposite one another. None of which have integer solutions for $2006 \leq M \leq 2045$.

7. There are 2006 nonzero real numbers written on a blackboard. An operation consists of choosing any two of these, a and b, erasing them, and writing $a + \frac{b}{2}$ and $b - \frac{a}{2}$ in their places. Prove that no sequence of operations can return the set of numbers to the original set.

   Solution: The sum of the squares of the two new numbers $(a + b/2)^2 + (b - a/2)^2 = a^2 + ab + b^2/4 + b^2 - ab + a^2/4 = 5a^2/4 + 5b^2/4$ is larger than the sum of the squares of the original numbers.

8. Is it possible to partition the set $N = \{1, 2, \ldots\}$ of positive integers into two element subsets $\{u, v\}$ such that for each integer $n \geq 1$, there is exactly one pair $\{u, v\}$ such that $|u - v| = n$?

Solution: Yes, it is possible. Start with the pair $\{1, 2\}$. If n doubleton sets $\{u_i, v_i\}, i = 1, 2, 3, \ldots, n$, where $u_i < v_i$ have been found, define the $n + 1^{\text{st}}$ subset $\{u_{n+1}, v_{n+1}\}$ as follows. Let u_{i+1} be the smallest positive integer that does not appear in any of the first n doubletons. Then $v_{n+1} = u_{n+1} + n + 1$. Note that $u_1 = 1 < u_2 < u_3 < \cdots < u_n < u_{n+1}$, and so the same can be said about the v_is, $v_1 = 2 < v_2 < v_3 < \cdots < v_n < v_{n+1}$. Since $v_i > u_i$, it follows that the set of u_is is disjoint from the set of v_is. By Mathematical Induction, for each n there is exactly one pair $\{u, v\}$ such that $|v - u| = n$.

Students familiar with Wythoff's game will recognize this problem. For example, see

`http://www.cut-the-knot.org/pythagoras/withoff.shtml`

9. Given a triangle ABC in the plane, prove that there is a line L in the plane that cuts the triangle into two polygons of equal area and equal perimeter.

Solution: Suppose the area of the triangle is a and the perimeter is p. For each X on the boundary of $\triangle ABC$, define $f(X)$ as follows:

(a) If $X \in \overline{AB}$, $f(X) = AX$,

(b) If $X \in \overline{BC}$, $f(X) = AB + BX$, and

(c) If $X \in \overline{CA}$, $f(X) = AB + BC + CX$.

Note the ambiguity $f(A) = 0$ and $f(A) = p$. For each $t \in [0, p/2]$, define $(X(t), Y(t))$ as follows:

(a) $X(t)$ and $Y(t)$ are points on the boundary of $\triangle ABC$,

(b) $f(X(t)) = t$, and

(c) $f(Y(t)) = t + p/2$.

Note that $X(0) = A$ and $Y(p/2) = A$. For each $t \in [0, p/2]$, the line segment $L = X(t)Y(t)$ divides $\triangle ABC$ into two parts with each part having the same perimeter. For each $t \in (0, p/2)$, define $A(t)$ to be the area of that part which contains the point A. Also, let $A(0)$ be the area of $\triangle AY(0)C$ and let $A(p/2)$ be the area of $\triangle ABX(p/2)$. Note that $Y(0) = X(p/2)$. Now $A(t)$ is a continuous function on $t \in [0, p/2]$. Also, $(A(0) + A(p/2))/2 = a/2$. Therefore, by the Intermediate Value Theorem, there exists $t \in [0, p/2]$ such that $A(t) = a/2$.

10. Suppose $(S, 0, +)$ is an Abelian group on the set S, and $(S, \cdot)$ is a binary operator on S. Also, suppose $(S, 0, +)$ distributes over $(S, \cdot)$. That is, $\forall a, b, x \in S, (a + x) \cdot (b + x) = (a \cdot b) + x$. Prove that if $(S, \cdot)$ is a group, then S is a singleton set.

    Solution: For all $x \in S$, $x \cdot x = (0 + x) \cdot (0 + x) = (0 \cdot 0) + x$. If $(S, \cdot)$ is a group with identity i, then $i \cdot i = (0 \cdot 0) + i$. Since $i \cdot i = i$, it follows that $0 \cdot 0 = 0$. Therefore, for all $x \in S, x \cdot x = x$. From this it follows that for all $x \in S, (x \cdot x)x^{-1} = x = i$, proving that S is a singleton.