2006 MATH Challenge

For full credit you must **justify** your answers.

1. Let $f(x) = (x+1)^2 e^{2x}$. The fiftieth derivative $f^{(50)}(0)$ can be expressed in the form $k2^n$ where k is an odd integer and n is a positive integer. Find k and n.

2. The three points $(4, 14, 8, 14)$, $(6, 6, 10, 8)$ and $(2, 4, 6, 8)$ are vertices of a 4-dimensional cube in 4-space. Find the center of the cube.

3. Compute $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 9}$.

4. Let $n \geq 1$ be fixed. Suppose n points are placed at random on a circle. Let $P(n)$ denote the probability that all n points lie on the same side of some diameter. Find $P(n)$. In particular, find $P(2)$ and $P(3)$.

5. Let $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ denote the set of nonzero decimal digits. Note that D has $\binom{9}{4} = 126$ four-element subsets. How many of these subsets $\{a, b, c, d\}$ can be used to build a three-digit base d number $N = abc_d$ such that the difference between N and the number $\overline{N}$ obtained by reversing the digits of N is a multiple of 21? For example $123_8 - 321_8 = (1 \cdot 8^2 + 2 \cdot 8 + 3) - (3 \cdot 8^2 + 2 \cdot 8 + 1) = -128 + 2 = -126 = -6(21)$, so the set $\{1, 2, 3, 8\}$ is one of the sets we need to count. Of course the number d used for the base must be the largest of the four numbers.

6. Suppose some faces of a large wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. Is it possible that the number of unit cubes with some red paint is exactly $M = 2006$ larger than the number of cubes with some black paint? Find the smallest number $M \geq 2006$ for which there is such a cube and find a way to paint the faces so that the number of unit cubes with some red paint is exactly M larger than the number of cubes with some black paint.

7. There are 2006 nonzero real numbers written on a blackboard. An operation consists of choosing any two of these, a and b, erasing them, and writing $a + \frac{b}{2}$ and $b - \frac{a}{2}$ in their places. Prove that no sequence of operations can return the set of numbers to the original set.

8. Is it possible to partition the set $N = \{1, 2, \ldots\}$ of positive integers into two element subsets $\{u, v\}$ such that for each integer $n \geq 1$, there is exactly one pair $\{u, v\}$ such that $|u - v| = n$?

9. Given a triangle ABC in the plane, prove that there is a line L in the plane that cuts the triangle into two polygons of equal area and equal perimeter.

10. Suppose $(S, 0, +)$ is an Abelian group on the set S, and $(S, \cdot)$ is a binary operator on S. Also, suppose $(S, 0, +)$ distributes over $(S, \cdot)$. That is, $\forall a, b, x \in S, (a + x) \cdot (b + x) = (a \cdot b) + x$. Prove that if $(S, \cdot)$ is a group, then S is a singleton set.